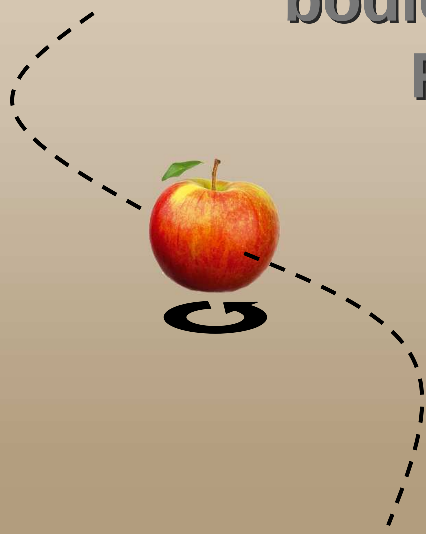
Motion of extended bodies in General Relativity



Dirk Puetzfeld
Center for applied space technology and microgravity
(ZARM, U Bremen)

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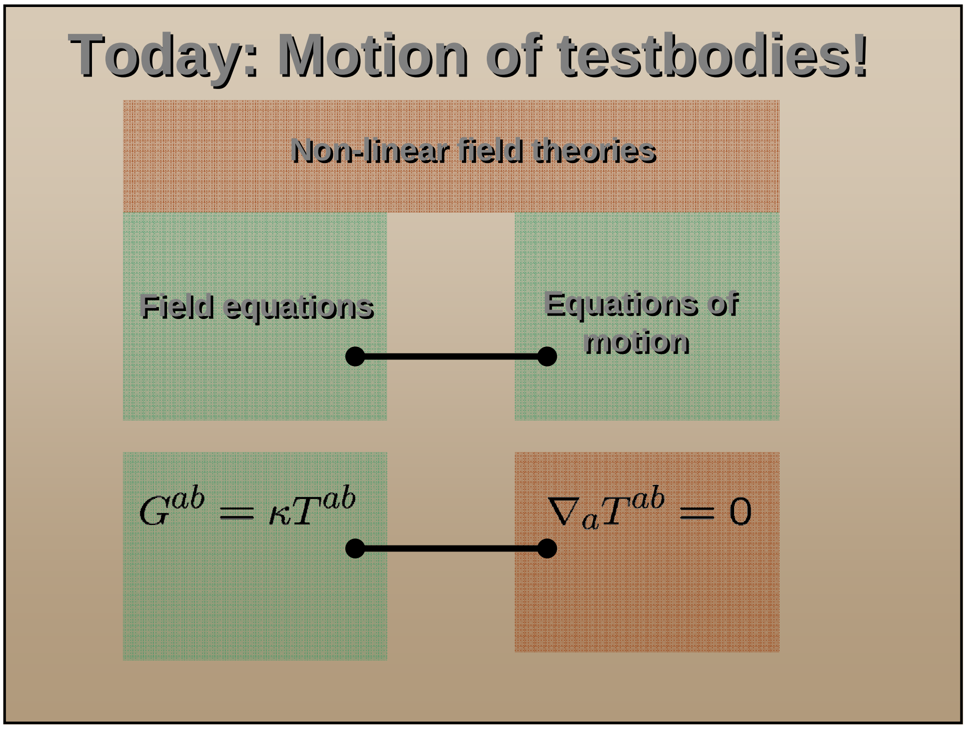
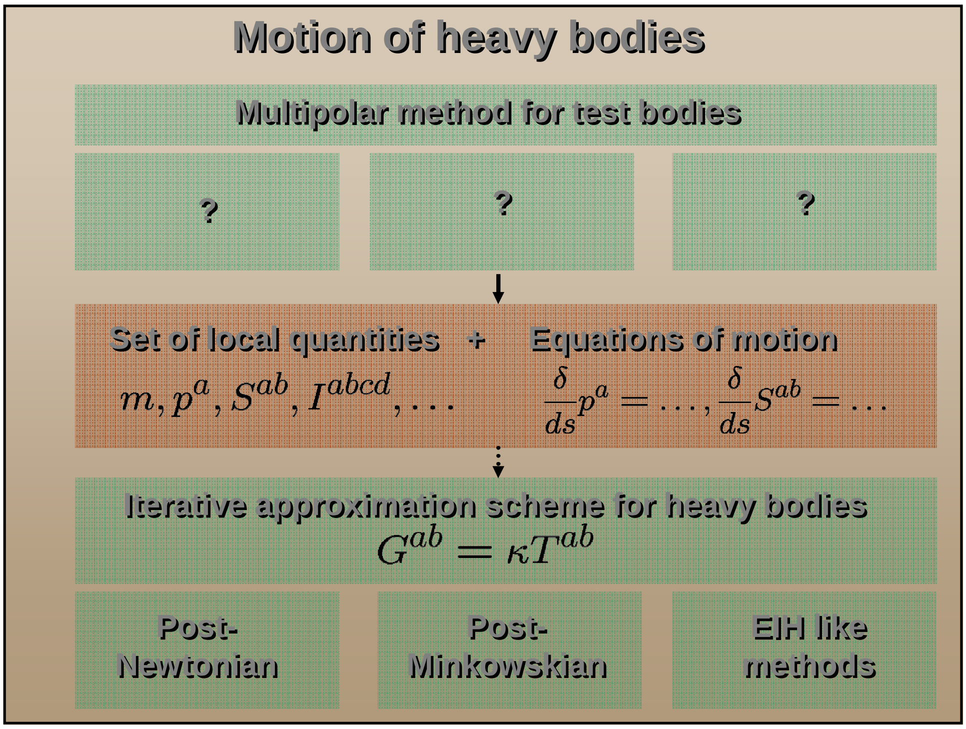
Outline

- Motivation
- Timeline
- Multipolar methods
- Open problems
- Outlook


Today: Motion of testbodies!
Non-linear field theories
Field equations
Equations of motion
$G^{ab} = \kappa T^{ab}$
$\nabla_a T^{ab} = 0$


Motion of heavy bodies
Multipolar method for test bodies
?
?
?
Set of local quantities + Equations of motion
$m, p^a, S^{ab}, I^{abcd}, \ldots$
$\frac{\delta}{ds} p^a = \ldots, \frac{\delta}{ds} S^{ab} = \ldots$
Iterative approximation scheme for heavy bodies
$G^{ab} = \kappa T^{ab}$
Post-Newtonian
Post-Minkowskian
EIH like methods

Genealogy: Multipole methods (early works)

Mathisson
1937

Shanmugadhasan
1945
1947

Tulczyjew
1959

Taub
1964
1965

Dixon
1970
1973
1974
1977
1979

Papapetrou
1951

Tulczyjew & Tulczyjew
1962

Dixon
1964

Beiglböck
1965

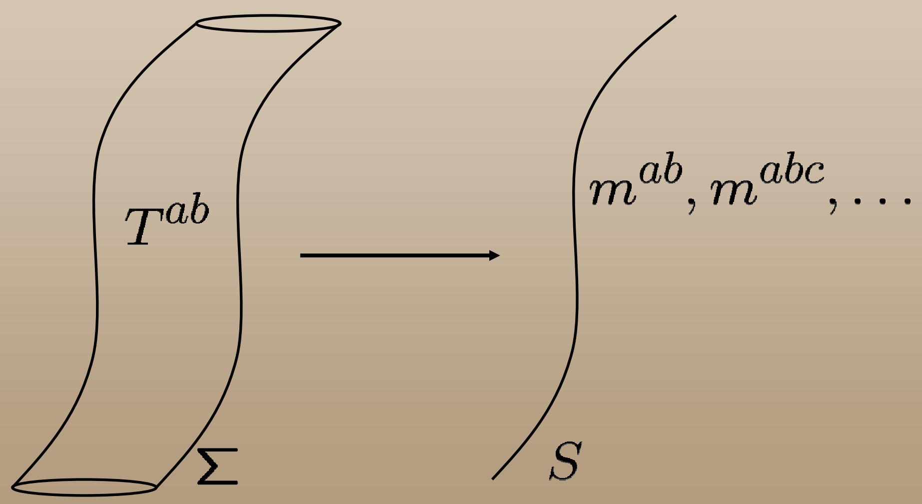
Madore
1969

Timeline: „Problem of motion“

Weyl	Einstein & Grommer Lanczos	Mathisson	EIH	Fock
1923	1927	1937	1938	1939

Infeld & Schild	Papapetrou	Tulczyjew	Infeld & Plebanski	Dixon
1949	1951	1959 1962	1949	1964 1967 1970 1973 1974 1977 1979 2008

Mathisson's method



Mathisson's method

$$\int_\Sigma T^{ab} p_{ab} d^4x = \int_S \left[p_{ab} m^{ab} + \left(\nabla_c p_{ab}\right) m^{cab} + \left(\nabla_c \nabla_d p_{ab}\right) m^{cdab} + \ldots \right] ds$$

``Gravitational skeleton" $m^{ab}, m^{cab}, m^{cdab}, \ldots$

$$\begin{aligned} p_{ab} &= \nabla_{(a} \xi_{b)} \\ \nabla_a T^{ab} &= 0 \end{aligned}$$

$$\int_S \left[\nabla_{(a} \xi_{b)} m^{ab} + \left(\nabla_c \nabla_{(a} \xi_{b)} \right) m^{cab} + \ldots \right] ds = 0$$

``Variational equation of mechanics"

Single-pole EOM

$$\int_S p_{ab} m^{ab} ds = \int_S m^{ab} \nabla_{(a}\xi_{b)} ds = 0$$

$$m^{ab} = \overset{0}{o}{}^{ab} + 2\, \overset{0}{o}{}^{(a} u^{b)} + \overset{0}{m}\, u^a u^b,$$

$$\overset{0}{o}{}^{a} := m^{cd} \rho_c^a u_d, \quad \overset{0}{o}{}^{ab} := m^{cd} \rho_c^a \rho_d^b, \quad \overset{0}{m} := m^{cd} u_c u_d$$

$$\overset{0}{o}{}^{ab} + \overset{0}{o}{}^{a} u^b = 0 \xrightarrow{\;|\cdot u_b\;} \overset{0}{o}{}^{a} = 0 \quad \text{and} \quad \overset{0}{o}{}^{ab} = 0$$

$$0 = \int_S \overset{0}{m}\, u^a u^b \nabla_a \xi_b ds = \int_S \left[u^a \nabla_a \left(\overset{0}{m}\, u^b \xi_b \right) - u^a \xi_b \nabla_a \left(\overset{0}{m}\, u^b \right) \right] ds$$

Single-pole EOM

$$\int_S p_{ab} m^{ab} ds = \int_S m^{ab} \nabla_{(a}\xi_{b)} ds = 0$$

$$m^{ab} = \overset{0}{o}{}^{ab} + 2\, \overset{0}{o}{}^{(a} u^{b)} + \overset{0}{m}\, u^a u^b,$$

$$\overset{0}{o}{}^{a} := m^{cd} \rho_c^a u_d, \quad \overset{0}{o}{}^{ab} := m^{cd} \rho_c^a \rho_d^b, \quad \overset{0}{m} := m^{cd} u_c u_d$$

$$\overset{0}{o}{}^{ab} + \overset{0}{o}{}^{a} u^b = 0 \xrightarrow{\;|\cdot u_b\;} \overset{0}{o}{}^{a} = 0 \quad \text{and} \quad \overset{0}{o}{}^{ab} = 0$$

$$0 = u^a \nabla_a \left(\overset{0}{m}\, u^b \right) = \frac{d}{ds} \left(\overset{0}{m}\, u^c \right) + \Gamma_{ab}{}^{c}\, \overset{0}{m}\, u^a u^b$$

Single-pole EOM

$$\int_S p_{ab} m^{ab} ds = \int_S m^{ab} \nabla_{(a}\xi_{b)} ds = 0$$

$$m^{ab} = \overset{0}{o}{}^{ab} + 2\,\overset{0}{o}{}^{(a}u^{b)} + \overset{0}{m}\, u^a u^b,$$

$$\overset{0}{o}{}^{a} := m^{cd}\rho^a_c u_d, \quad \overset{0}{o}{}^{ab} := m^{cd}\rho^a_c \rho^b_d, \quad \overset{0}{m} := m^{cd} u_c u_d$$

$$\overset{0}{o}{}^{ab} + \overset{0}{o}{}^{a} u^b = 0 \quad \xrightarrow{|\cdot u_b} \quad \overset{0}{o}{}^{a} = 0 \quad \text{and} \quad \overset{0}{o}{}^{ab} = 0$$

$|\cdot u_b$ and $u^a u_a = 1$

$$d\overset{0}{m}/ds = 0 \qquad \frac{du^c}{ds} + \Gamma_{ab}{}^c u^a u^b = 0$$

Single-Pole EOM

Pole-dipole EOM

$$\int_S \left[p_{ab} m^{ab} + (\nabla_c p_{ab})\, m^{cab} \right] ds = 0$$

$$M \frac{Du^a}{Ds} - K^a{}_{bcd} u^b n^{cd} + 2n^{ad} \frac{D^2 u_d}{Ds^2} = 0 \; (1)$$

$$\frac{Dn^{bc}}{Ds} - n^{ca} \frac{Du_a}{Ds} u^b + n^{ba} \frac{Du_a}{Ds} u^c = 0 \qquad (2)$$

$$M = \text{const} \qquad n^{ab} = -n^{ba} \qquad (3)$$

$$m^{abc} = m^{\hat a \hat b \hat c} + m^{\hat a \hat b f} u^c u_f + m^{\hat a f \hat c} u^b u_f + m^{\hat a e f} u^b u^c u_e u_f$$

$$^* m^{abc} = m^{\hat a \hat b \hat c}, \quad n^{ba} = m^{\hat a \hat b f} u_f, \quad n^{ca} = m^{\hat a f \hat c} u_f, \quad n^a = m^{\hat a e f} u_e u_f$$

Tulczyjew (1959)

$$\tilde{T}^{ab} = \int_{-\infty}^{+\infty} \left\{ t^{ab}\delta_{(4)} + \nabla_c \left[t^{cab}\delta_{(4)} \right] + \nabla_d \nabla_c \left[t^{dcab}\delta_{(4)} \right] + \ldots \right\} ds$$

$$\tilde{A}^{b_1 \ldots b_n} = \sum_{k=0}^{m} \int_{-\infty}^{\infty} \nabla_{c_1 \ldots c_k} \left[\alpha^{c_1 \ldots c_k b_1 \ldots b_n} \delta_{(4)} \left(x^a - Y^a \right) \right]$$

$$\alpha^{c_1 \ldots c_k b_1 \ldots b_n} = \alpha^{(c_1 \ldots c_k) b_1 \ldots b_n}$$

$$u_{c_1} \alpha^{c_1 \ldots c_k b_1 \ldots b_n} = 0$$

Canonical form

$$\int_D \tilde{A}^{b_1 \ldots b_n} p_{b_1 \ldots b_n} = 0 \longrightarrow \alpha^{c_1 \ldots c_k b_1 \ldots b_n} = 0$$

``Generalized du Bois-Reymond"

Equations of motion for pole-dipole particles

$$0 = \frac{\delta}{ds}\left(u_d \frac{\delta}{ds} \overset{1}{t}{}^{db} + \overset{0}{o}{}^{b} + \overset{0}{t}\, u^b - \dot{u}_c \overset{1}{o}{}^{cb} - \dot{u}_c \overset{1}{o}{}^{c} u^b \right)$$

$$+ \frac{1}{2} R_{ace}{}^{b} \left[2u^a \left(\overset{1}{o}{}^{ce} + \overset{1}{o}{}^{c} u^e \right) + \overset{1}{o}{}^{cae} + \overset{1}{o}{}^{ca} u^e \right] \qquad \text{E}$$

$$\overset{0}{o}{}^{a} = -u_d \rho^a_c \frac{\delta}{ds} \left(\overset{1}{o}{}^{cd} + \overset{1}{o}{}^{c} u^d + \overset{1}{t}{}^{cd} \right) \qquad \text{C}$$

$$\overset{0}{o}{}^{ab} = -\rho^b_d \rho^a_c \frac{\delta}{ds} \left(\overset{1}{o}{}^{cd} + \overset{1}{o}{}^{c} u^d + \overset{1}{t}{}^{cd} \right) \qquad \text{E+C}$$

$$\overset{1}{o}{}^{(ca)} = 0 \qquad \text{C}$$

$$\overset{1}{o}{}^{(ca)b} = 0 \qquad \text{C}$$

Steinhoff & Puetzfeld 2010

$$t^{ab} = \overset{0}{o}{}^{ab} + 2\, \overset{0}{o}{}^{(a} u^{b)} + \overset{0}{t}\, u^a u^b$$

$$t^{abc} = \overset{1}{o}{}^{abc} + 2\, \overset{1}{o}{}^{a(b} u^{c)} + \overset{1}{o}{}^{a} u^b u^c + u^a\, \overset{1}{t}{}^{bc}$$

Equations of motion for pole-dipole particles

$$\frac{\delta}{ds}\overset{1}{p}{}^{b} + \frac{1}{2}u^{e}\,\overset{1}{S}{}^{ac}R_{ace}{}^{b} = 0$$

$$\frac{\delta\,\overset{1}{S}{}^{ab}}{ds} - u^{a}u_{c}\frac{\delta\,\overset{1}{S}{}^{cb}}{ds} - u^{b}u_{c}\frac{\delta\,\overset{1}{S}{}^{ac}}{ds} = 0$$

Steinhoff & Puetzfeld 2010

$$\begin{aligned}
\overset{1}{S}{}^{ab} &:= -2\left(\overset{1}{o}{}^{[ab]} + \overset{1}{o}{}^{[a}u^{b]}\right)\\
\overset{1}{p}{}^{b} &:= \left(\overset{0}{t} - u_{c}\dot{u}_{d}\,\overset{1}{S}{}^{cd} + u_{c}u_{d}\frac{\delta}{ds}\,\overset{1}{t}{}^{cd}\right)u^{b} + u_{d}\frac{\delta}{ds}\,\overset{1}{S}{}^{bd}\\
&= \overset{1}{m}\,u^{b} + u_{d}\frac{\delta}{ds}\,\overset{1}{S}{}^{bd}
\end{aligned}$$

$$\begin{aligned}
t^{ab} &= \overset{0}{o}{}^{ab} + 2\,\overset{0}{o}{}^{(a}u^{b)} + \overset{0}{t}\,u^{a}u^{b}\\
t^{abc} &= \overset{1}{o}{}^{abc} + 2\,\overset{1}{o}{}^{a(b}u^{c)} + \overset{1}{o}{}^{a}u^{b}u^{c} + u^{a}\,\overset{1}{t}{}^{bc}
\end{aligned}$$

„Combined quantities" at different orders

Mass $\overset{0}{m} := \overset{0}{t}$

Momentum $\overset{0}{p}{}^{b} := \overset{0}{m}\,u^{b}$

Pole

Mass $\overset{1}{m} := \overset{0}{t} - u_{c}\dot{u}_{d}\,\overset{1}{S}{}^{cd} + u_{c}u_{d}\frac{\delta}{ds}\,\overset{1}{t}{}^{cd}$

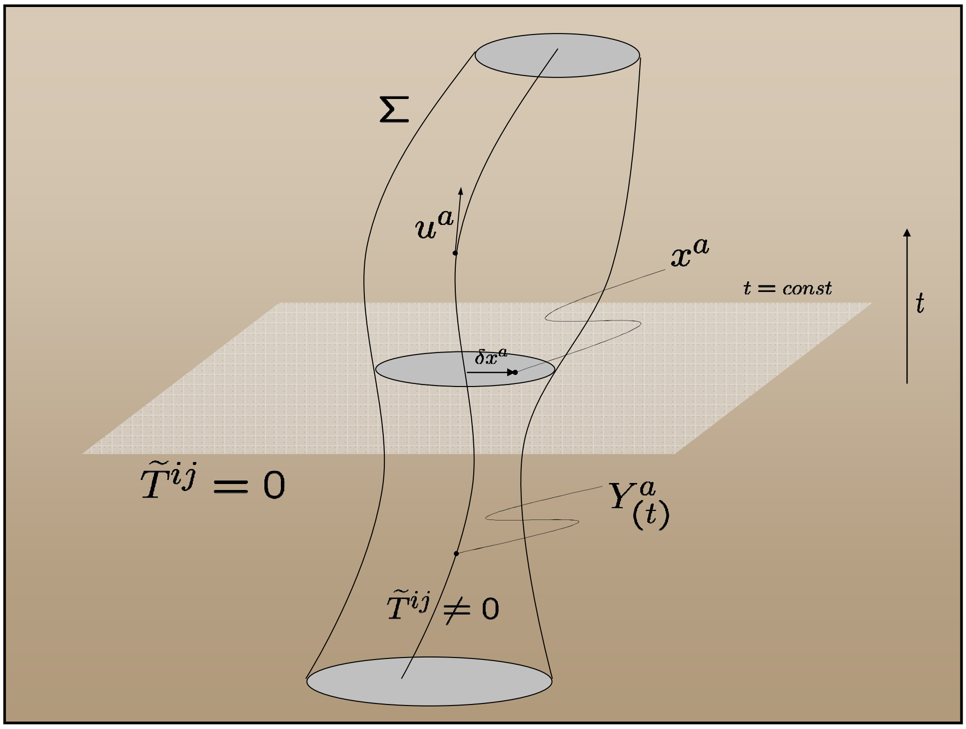
Momentum $\overset{1}{p}{}^{b} := \overset{1}{m}\,u^{b} + u_{d}\frac{\delta}{ds}\,\overset{1}{S}{}^{bd}$

Spin $\overset{1}{S}{}^{ab} := -2\left(\overset{1}{o}{}^{[ab]} + \overset{1}{o}{}^{[a}u^{b]}\right)$

Pole-Dipole

$$\begin{aligned}
t^{ab} &= \overset{0}{o}{}^{ab} + 2\,\overset{0}{o}{}^{(a}u^{b)} + \overset{0}{t}\,u^{a}u^{b}\\
t^{abc} &= \overset{1}{o}{}^{abc} + 2\,\overset{1}{o}{}^{a(b}u^{c)} + \overset{1}{o}{}^{a}u^{b}u^{c} + u^{a}\,\overset{1}{t}{}^{bc}
\end{aligned}$$

Steinhoff & Puetzfeld 2010



$$\nabla_a T^{ab} = 0$$

$$\Gamma_{ij}{}^{k}\big|_x = \Gamma_{ij}{}^{k}\big|_Y + \delta x^a\, \Gamma_{ij}{}^{k}{}_{,a}\big|_Y + \frac{1}{2}\delta x^a \delta x^b\, \Gamma_{ij}{}^{k}{}_{,ab}\big|_Y + \ldots$$

$$\frac{d}{dt}\int\left(\prod_{\alpha=1}^{n}\delta x^{b_\alpha}\right)\tilde{T}^{i0} = \sum_{\beta=1}^{n}\left[\int\left(\prod_{\alpha=1,\alpha\neq\beta}^{n}\delta x^{b_\alpha}\right)\tilde{T}^{ib_\beta} - v^{b_\beta}\int\left(\prod_{\alpha=1,\alpha\neq\beta}^{n}\delta x^{b_\alpha}\right)\tilde{T}^{i0}\right] - \int\left(\prod_{\alpha=1}^{n}\delta x^{b_\alpha}\right)\left(\overset{\{\}}{\Gamma}{}_{kj}{}^{i}\tilde{T}^{kj}\right)$$

$$\overline{T}^{b_1\cdots b_n ij} := \int\left(\prod_{\alpha=1}^{n}\delta x^{b_\alpha}\right)\tilde{T}^{ij}$$

$$\delta x^a = x^a - Y^a$$
$$\rho^b{}_a = \delta x^b{}_{,a} = \delta^b_a - v^b\delta^0_a = \delta^b_a - \delta^b_0\delta^0_a = \delta^b_\alpha\delta^\alpha_a$$

$$\frac{d}{dt}\overline{T}^{b_1\cdots b_n i0} = \sum_{\beta=1}^{n}\left(\overline{T}^{b_1\cdots\breve{b}_\beta\cdots b_n i b_\beta} - v^{b_\beta}\overline{T}^{b_1\cdots\breve{b}_\beta\cdots b_n i0}\right) - \int\left(\prod_{\alpha=1}^{n}\delta x^{b_\alpha}\right)\left(\overset{\{\}}{\Gamma}{}_{kj}{}^{i}\tilde{T}^{kj}\right)$$

Equations of motion for pole-dipole particles

$$
\begin{aligned}
\frac{D}{Ds}\left(m u^a + u_b \frac{DS^{ab}}{Ds}\right) + \frac{1}{2} S^{cd} u^b R^a{}_{bcd} &= 0 \\
\frac{DS^{ab}}{Ds} + u^a u_c \frac{DS^{bc}}{Ds} - u^b u_c \frac{DS^{ac}}{Ds} &= 0
\end{aligned}
$$

Papapetrou 1951

$$S^{ab} := \overline{T}^{ab0} - \overline{T}^{ba0} \quad M^{ab} := u^0 \overline{T}^{ab} \quad m := \frac{1}{u^0}\left(M^{a0} + \Gamma_{bc}{}^a S^{b0} u^c\right) u_a$$

$$\overline{T}^{b_1 \cdots b_n ij} := \int \left(\prod_{\alpha=1}^{n} \delta x^{b_\alpha}\right) \tilde{T}^{ij} \qquad \delta x^a := x^a - Y^a$$

Mathisson

i. Covariant ansatz (!)

ii. Original method based on linearized background

iii. Supplementary condition imposed at early stage in original work

iv. Difficult interpretation of the resulting moments at higher orders

v. A priori <u>no</u> integral representation of the moments

vi. Tedious <u>but</u> systematic to carry out at quadrupole and higher orders

Papapetrou

i. Non-covariant ansatz

ii. Requires tricky introduction of new objects to achieve covariance

iii. Supplementary condition completely free

iv. Difficult interpretation of the resulting moments at higher orders

v. A priori integral representation of the moments

vi. Tedious to carry out at quadrupole and higher orders

Quadrupolar testbodies

$$\widetilde{T}^{ab} = \int \mathrm{t}^{ab}\delta_{(4)} + \int \nabla_c \left[\mathrm{t}^{cab}\delta_{(4)}\right] + \int \nabla_d \nabla_c \left[\mathrm{t}^{dcab}\delta_{(4)}\right]$$

„Skeleton“ for quadrupolar testbodies

$$\mathrm{t}^{ab} = \overset{0}{n}{}^{ab} + 2\,\overset{0}{n}{}^{(a}u^{b)} + \overset{0}{n}\,u^a u^b$$

$$\mathrm{t}^{cab} = \overset{1}{n}{}^{cab} + 2\,\overset{1}{n}{}^{c(a}u^{b)} + \overset{1}{n}{}^{c}u^a u^b$$

$$\mathrm{t}^{dcab} = \overset{2}{n}{}^{dcab} + 2\,\overset{2}{n}{}^{dc(a}u^{b)} + \overset{2}{n}{}^{dc}u^a u^b$$

Canonical form

$$\nabla_a \widetilde{T}^{ab} = 0$$

↓

3 $\int \nabla_a \nabla_d \nabla_c \left\{\mathrm{t}^{(\hat{d}\hat{c}\hat{a})b}\delta_{(4)}\right\}$

2 $+\int \nabla_d \nabla_c \left\{\left[\rho_e^{(c}\rho_f^{d)}\frac{\delta}{ds}\left[\mathrm{t}^{\hat{f}\hat{e}ab}u_a\right] + \mathrm{t}^{(\hat{c}\hat{d})b}\right]\delta_{(4)}\right\}$

1 $+\int \nabla_d \left\{\left[\rho_e^d \frac{\delta}{ds}\left(\mathrm{t}^{eab}u_a - 2\dot{u}_c \mathrm{t}^{\hat{e}\hat{c}ab}u_a\right) + R_{acf}{}^b\left(\mathrm{t}^{\hat{d}\hat{c}\hat{a}f} + 2u^a \mathrm{t}^{\hat{d}\hat{c}ef}u_e\right)\right.\right.$
$\left.\left. + R_{acf}{}^{\hat{d}}\left(\frac{1}{3}\mathrm{t}^{\hat{f}\hat{c}\hat{a}b} + u^a \mathrm{t}^{\hat{f}\hat{c}eb}u_e\right) + \mathrm{t}^{\hat{d}b}\right]\delta_{(4)}\right\}$

0 $+\int \left\{\left[\frac{\delta}{ds}\left(R_{acf}{}^d u^a u_d \mathrm{t}^{\hat{f}\hat{c}eb}u_e + \frac{1}{3}R_{ade}{}^c u_c \mathrm{t}^{\hat{e}\hat{d}\hat{a}b} + 2\dot{u}_d \dot{u}_c \mathrm{t}^{\hat{d}\hat{c}ab}u_a + \mathrm{t}^{cb}u_c - \dot{u}_c u_d \mathrm{t}^{cdb}\right)\right.\right.$
$+\frac{2}{3}R_{dae}{}^b{}_{;c}\mathrm{t}^{\hat{c}\hat{d}\hat{a}e} - R_{acf}{}^b{}_{;d}u^a \mathrm{t}^{\hat{c}\hat{d}ef}u_e - R_{dce}{}^b u^d \dot{u}_f \mathrm{t}^{\hat{f}\hat{c}ae}u_a$
$\left.\left. +\frac{1}{2}R_{ace}{}^b\left(2\mathrm{t}^{\hat{c}de}u^a u_d + \mathrm{t}^{\hat{c}\hat{a}e}\right)\right]\delta_{(4)}\right\} = 0$

Steinhoff & Puetzfeld 2009

Quadrupolar testbodies

$$\rho^a_c \rho^b_d \frac{\delta}{ds} \overset{2}{S}{}^{cd} = \rho^{[a}_g \rho^{b]}_e R_{dcf}{}^g \left[\frac{8}{3} \overset{2}{n}{}^{edcf} + 8 \overset{2}{n}{}^{edf} u^c + 4 \overset{2}{n}{}^{ed} u^f u^c \right]$$

$$\frac{\delta}{ds} \overset{2}{p}{}^{b} = -\frac{1}{2} R_{ace}{}^b u^e \overset{2}{S}{}^{ac} + \nabla_d R_{ace}{}^b \left[\frac{2}{3} \overset{2}{n}{}^{dcae} + \frac{4}{3} \overset{2}{n}{}^{dce} u^a + \frac{4}{3} \overset{2}{n}{}^{aed} u^c + \overset{2}{n}{}^{dc} u^e u^a + \overset{2}{n}{}^{ae} u^c u^d \right]$$

Steinhoff & Puetzfeld 2010

Quadrupolar EOM (evolution type)

$$\overset{2}{S}{}^{ab} := -2 \left(\overset{1}{n}{}^{[ab]} + \overset{1}{n}{}^{[a} u^{b]} - 2 (\overset{2}{n}{}^{c[ab]} + \overset{2}{n}{}^{c[a} u^{b]}) \dot{u}_c \right)$$

$$\overset{2}{m} := \overset{0}{n} + \overset{2}{S}{}^{ac} \dot{u}_a u_c - \frac{2}{3} R_{ace}{}^d \overset{2}{n}{}^{eca} u_d$$

$$\overset{2}{p}{}^{b} := \overset{2}{m} u^b + u_a \frac{\delta}{ds} \overset{2}{S}{}^{ba} + R_{ace}{}^d \left(\frac{4}{3} \overset{2}{n}{}^{abce} u_d + 4 \overset{2}{n}{}^{bae} u^c u_d + \frac{4}{3} \overset{2}{n}{}^{aec} \rho^b_d + 2 \overset{2}{n}{}^{ae} u^c \rho^b_d \right)$$

Quadrupolar testbodies

$$\frac{\delta}{ds} S^{ab} = 2 p^{[a} u^{b]} + \frac{4}{3} R_{cde}{}^{[a} I^{b]cde}$$

$$\frac{\delta}{ds} p_a = \frac{1}{2} R_{abcd} u^b S^{cd} - \frac{1}{6} \nabla_a R_{bcde} I^{bedc}$$

Steinhoff & Puetzfeld 2010

Quadrupolar EOM (compact form)

$$S^{ab} \equiv \overset{2}{S}{}^{ab}, \qquad m \equiv \overset{2}{m}, \qquad p^a \equiv \overset{2}{p}{}^{a}$$

$$I^{dcab} := 2 \left(\overset{2}{n}{}^{dcab} + 2 \overset{2}{n}{}^{dc(a} u^{b)} + 2 \overset{2}{n}{}^{ab(d} u^{c)} + \overset{2}{n}{}^{dc} u^a u^b + \overset{2}{n}{}^{ab} u^d u^c - 2 u^{(d} \overset{2}{n}{}^{c)(a} u^{b)} \right)$$

$$I^{abcd} = I^{(ab)(cd)} = I^{cdab}$$

$$I^{(abc)d} = 0 \quad \Leftrightarrow \quad I^{abcd} + I^{bcad} + I^{cabd} = 0$$

Quadrupolar testbodies

$$\begin{aligned}\widetilde{T}^{ab} &= \int \left(u^{(a}p^{b)} + \frac{1}{3} R_{cde}{}^{(a} I^{b)cde}\right)\delta_{(4)} \\ &\quad - \int \nabla_c \left(S^{c(a}u^{b)}\delta_{(4)}\right) + \frac{1}{2}\int \nabla_d\nabla_c\left(I^{dcab}\delta_{(4)}\right)\end{aligned}$$

Steinhoff & Puetzfeld 2010

EM tensor for quadrupolar testbodies

$$S^{ab} \equiv \overset{2}{S}{}^{ab}, \qquad m \equiv \overset{2}{m}, \qquad p^a \equiv \overset{2}{p}{}^{a}$$

$$\begin{aligned}I^{dcab} &:= 2\Big(\overset{2}{n}{}^{dcab} + 2\,\overset{2}{n}{}^{dc(a}u^{b)} + 2\,\overset{2}{n}{}^{ab(d}u^{c)} \\ &\quad + \overset{2}{n}{}^{dc}u^a u^b + \overset{2}{n}{}^{ab}u^d u^c - 2u^{(d}\,\overset{2}{n}{}^{c)(a}u^{b)}\Big) \\ I^{abcd} &= I^{(ab)(cd)} = I^{cdab} \\ I^{(abc)d} &= 0 \quad \Leftrightarrow \quad I^{abcd} + I^{bcad} + I^{cabd} = 0\end{aligned}$$

Dixon‘s approach

$$\int_\Sigma T^{ab}p_{ab}d^4x = \int_S \left[p_{ab}m^{ab} + (\nabla_c p_{ab})\, m^{cab} + (\nabla_c\nabla_d p_{ab})\, m^{cdab} + \ldots\right] ds$$

↓

$$\int_\Sigma T^{\alpha\beta}p_{\alpha\beta}d^4x = M^{\alpha\beta}\left[\Phi_{\alpha\beta} + \frac{1}{2}\Lambda^\gamma \nabla_{*\{\alpha}G_{\gamma\beta\}}\right]$$

↓

$$M^{\alpha\beta}\left[\nabla_{*(\alpha}M_{\beta)} + \Xi_{\alpha\beta}\right] = 0$$

``Modified variational equation of mechanics“

Dixon‘s approach

$$\begin{aligned} \frac{\delta}{ds} p^{\alpha} &= \frac{1}{2} v^{\beta} S^{\gamma\delta} R^{\alpha}{}_{\beta\gamma\delta} + F^{\alpha} \\ \frac{\delta}{ds} S^{\alpha\beta} &= 2 p^{[\alpha} v^{\beta]} + L^{\alpha\beta} \end{aligned}$$

Dixon 1974

$$\begin{aligned} p^{\kappa} &= - \int_{\Sigma(s)} \overset{-1}{\sigma}{}^{\kappa}{}_{\gamma} \sigma^{\gamma}{}_{\alpha} \tilde{T}^{\alpha\beta} d\Sigma_{\beta} \\ S^{\kappa\lambda} &= -2 \int_{\Sigma(s)} \sigma^{[\lambda} \overset{-1}{\sigma}{}_{\alpha}{}^{\kappa]} \tilde{T}^{\alpha\beta} d\Sigma_{\beta} \end{aligned}$$

Dixon‘s approach

$$\begin{aligned} F_{\alpha} &:= \frac{1}{2} \sum_{n \geq 2} \frac{1}{n!} m^{\epsilon \cdots \delta\gamma\beta} \nabla_{\alpha} g_{\gamma\beta,\epsilon\cdots\delta} \\ L^{\alpha\beta} &:= \sum_{n \geq 1} \frac{1}{n!} g^{\gamma[\alpha} m^{\beta]\epsilon\cdots\delta\zeta\eta} g_{\{\gamma\eta,\zeta\}\epsilon\cdots\delta}, \end{aligned}$$

$$g_{\alpha\beta,\gamma\delta} = -\frac{2}{3} R_{\alpha(\gamma\delta)\beta}, \quad g_{\alpha\beta,\gamma\delta\epsilon} = -\nabla_{(\gamma} R_{|\alpha|\delta\epsilon)\beta}, \quad \ldots$$

$$\begin{aligned} F_{\alpha} &= -\frac{1}{6} m^{\epsilon\delta\gamma\beta} \nabla_{\alpha} R_{\gamma(\epsilon\delta)\beta} - \frac{1}{12} m^{\epsilon\delta\kappa\gamma\beta} \nabla_{\alpha} \left[\nabla_{(\epsilon} R_{|\gamma|\delta\kappa)\beta} \right] + \ldots \\ L^{\alpha\beta} &= -\frac{2}{3} g^{\gamma[\alpha} m^{\beta]\epsilon\zeta\eta} \left[R_{\gamma(\zeta\epsilon)\eta} - R_{\eta(\gamma\epsilon)\zeta} + R_{\zeta(\eta\epsilon)\gamma} \right] \\ & \quad -\frac{1}{2} g^{\gamma[\alpha} m^{\beta]\epsilon\delta\zeta\eta} \left[\nabla_{(\zeta} R_{|\gamma|\epsilon\delta)\eta} - \nabla_{(\gamma} R_{|\eta|\epsilon\delta)\zeta} + \nabla_{(\eta} R_{|\zeta|\epsilon\delta)\gamma} \right] + \ldots \end{aligned}$$

Motion of heavy bodies

Multipolar method for test bodies

Mathisson Tulczyjew	Papapetrou	Dixon

↓

Set of local quantities + Equations of motion

$m, p^a, S^{ab}, I^{abcd}, \dots$ $\qquad$ $\frac{\delta}{ds} p^a = \dots, \frac{\delta}{ds} S^{ab} = \dots$

↓

Iterative approximation scheme for heavy bodies

$$G^{ab} = \kappa T^{ab}$$

Post-Newtonian	Post-Minkowskian	EIH like methods

Open problems

- Choice / interpretation of moments
- Form of the equations at higher (all) orders
- Supplementary condition(s)
- Conserved quantities (at quadrupolar and higher orders)
- Radiation reaction / description of heavy bodies
- Relation to other approximation schemes
- Experimental confirmation