Integrability of five dimensional gravity theories and inverse scattering construction of dipole black rings

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based on:

arXiv:0912.3199 with P. Figueras, E. Jamsin and A. Virmani (published in CQG 27 (2010));
arXiv:1108.3527 with M. J. Rodriguez and A. Virmani (published in JHEP 11 (2011)).

The non-trivial pursuit for exact black hole solutions

- Exact solutions are precioussss. . . but hard to find.
- Gravity in higher dimensions has attracted much attention in recent years. [Emparan& Reall, 2008]

  Some reasons to study higher dimensional black holes:

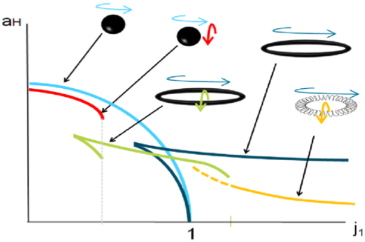
  AdS/CFT String theory TeV-scale gravity

- Gravity in higher dimensions is much richer. . . and much harder to find exact solutions.

The black hole bestiary

How much do we know about stationary, axisymmetric, asymptotically flat black hole solutions in general D dimensions?

- In 4D the list is exhausted by the Kerr-Newman solution.
- In 5D much progress has been achieved over the last decade.



from [Rodriguez, 2010]

- In D$\geq$ 6 exact solutions get scarcer...

$\Rightarrow$ This situation calls for solution-generating techniques.

Solution-generating techniques

It often happens when one is trying to solve an equation that an algorithm will exist for constructing new solutions from a given solution. [Wald, 1984]

- The first solution-generating technique applied to gravity was due to Ehlers [Ehlers, 1957] and later developed by Geroch. [Geroch, 1972]
- A class of solution-generating techniques is provided by hidden symmetries of dimensionally reduced theories down to 3D.
  [Ehlers, 1959] [Ernst, 1968] [Neugebauer & Kramer, 1969] [Kinnersley, 1973] [Maison, 1979]
- The so-called Bäcklund transformations were developed in the late 1870's.
  [Harrison, 1978] [Neugebauer, 1979,1980] [Hoenselaers, Kinnersley & Xanthopoulos, 1979]
- The inverse scattering method (ISM) appeared simultaneously. [Belinsky & Zakharov, 1978, 1979] This is a systematic procedure for generating new solutions from known solutions.

  An improvement was made more recently, making the ISM more powerful. [Pomeransky, 2006]

Solution-generating techniques

- Various (super)gravity theories in D dimensions can be reduced (in the presence of $D-2$ commuting Killing symmetries) to 2D dilaton-gravity coupled to a non-linear σ-model living on a coset space.

- Such 2D coset models are known to be completely integrable.
  [Maison, 1978] [Belinsky & Zakharov, 1978, 1979] [Breitenlohner & Maison, 1987] [Nicolai, 1991]

- Integrability of these models had not been used as a solution-generating technique.

  The only exceptions: — vacuum gravity in various dimensions [Belinsky & Zakharov, 1978, 1979]
  — 4D Einstein-Maxwell theory [Alekseev, 1981]

Goal and motivation: part I

- Goal: Explore the integrability of 5D minimal SUGRA by applying the ISM.

  Previous formulations were not well suited for ISM. [Breitenlohner & Maison, 1987] [Nicolai, 1991]

- Motivation: For 5D minimal SUGRA, a 5-parameter (mass, two angular momenta, electric charge and dipole charge) family of black ring solutions should exist. [Elvang et al. 2005]

  However, at present all known regular black rings have no more than 3 parameters.

Goal and motivation: part II

- Goal: Construct a dipole black ring in 5D Einstein-Maxwell-dilaton using the ISM (in 6D).
- Motivation: Same as before, *plus*

  — It is not understood how to systematically generate dipole charges.
  The original dipole black ring solution was constructed using educated guesswork. [Emparan, 2004]

  — There is also an algorithmic construction of a dipole black ring solution.
  However, it cannot generate multiple rotations... [Yazadjiev, 2006]

  — The ISM is sufficiently robust to deal with multiple rotations.

Outline

Outline

Canonical form of the metric

- Consider stationary, axisymmetric solutions of Einstein eqs. in vacuum: $R_{\mu\nu} = 0$.
- Assume $D-2$ commuting Killing vector fields.
  Then metric can be written in canonical form: [Wald, 1984] [Emparan & Reall, 2002] [Harmark, 2004]

$$ds^2 = \sum_{i,j=0}^{D-3} G_{ij} dx^i dx^j + e^{2\nu} \left[d\rho^2 + dz^2 \right] , \qquad \det G = -\rho^2 ,$$

  where all metric components depend only on (ρ, z).

- The vacuum Einstein equations divide into two groups.

$$\text{For } G_{ij} : \quad \partial_\rho U + \partial_z V = 0 , \qquad \text{where} \quad U \equiv \rho(\partial_\rho G)G^{-1}, \quad V \equiv \rho(\partial_z G)G^{-1} .$$

$$\text{For } \nu : \quad \partial_\rho \nu = -\frac{1}{2\rho} + \frac{1}{8\rho}\mathrm{Tr}(U^2 - V^2) , \qquad \partial_z \nu = \frac{1}{4\rho}\mathrm{Tr}(UV) .$$

- Integrability condition $\partial_\rho\partial_z\nu = \partial_z\partial_\rho\nu$ is automatically satisfied $\rightarrow$ Focus on G_{ij}.

Static, axisymmetric solutions

- It is easy to construct static (diagonal) solutions. [Weyl, 1917] [Emparan & Reall, 2002]
  Writing the Killing part of the metric as

$$G = \mathrm{diag}\{-e^{2U_0}, e^{2U_1}, e^{2U_2}, \ldots\},$$

  the functions U_i are simply (axisymmetric) harmonic functions in 3D flat space, subject to a constraint:

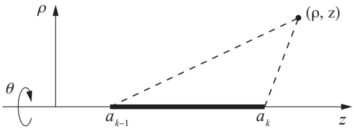
$$\nabla^2 U_i = 0\,, \qquad \sum_{i=0}^{D-3} U_i = \log\rho\,.$$

- Boundary conditions must be specified on the z-axis.
  They correspond to sources in the form of zero-thickness rods.
- The U_i's can be regarded as Newtonian potentials produced by these sources.
- Meaning of the constraint: sources must add up to give an infinite rod.
- All rods have constant mass density $\eta = 1/2$ in units where $G_N = 1$.
  (Other choices lead to curvature singularities.)

Static, axisymmetric solutions

So, what do potentials produced by zero-thickness rods look like?

- For an infinite rod along the z-axis: $U_i = \eta \log \rho^2$
- For a semi-infinite rod $[a_k, +\infty)$: $U_i = \eta \log \mu_k$
- For a semi-infinite rod $(-\infty, a_k]$: $U_i = \eta \log(\rho^2/\mu_k) = \eta \log(-\overline{\mu}_k)$
- For a finite rod $[a_{k-1}, a_k]$: $U_i = \eta \log(\mu_{k-1}/\mu_k)$



The potentials are entirely specified by the location of the rod endpoints, a_k.
These appear in combinations known as solitons and anti-solitons:

$$\mu_k = \sqrt{\rho^2 + (z - a_k)^2} - (z - a_k)\,, \qquad \overline{\mu_k} = -\sqrt{\rho^2 + (z - a_k)^2} - (z - a_k)\,,$$

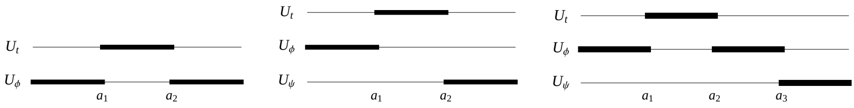
which satisfy $\mu_k \overline{\mu_k} = -\rho^2$.

Static, axisymmetric solutions

Conclusion

Static vacuum solutions of the Einstein equations with $D-2$ commuting Killing vector fields are completely determined by rod-like sources, only subject to the constraint that they must add up to produce an infinite rod. [Emparan & Reall, 2002]

- Some examples:



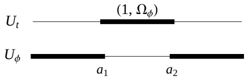
4D Schwarzschild | 5D Tangherlini | static black ring

finite timelike rods $\longrightarrow$ event horizons
semi-infinite spacelike rods $\longrightarrow$ axes of rotation

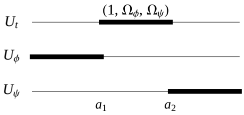
- Downside: the general class of metrics encompassed by this program does *not* include asymptotically flat solutions if $D \geq 6$ . . .

Stationary, axisymmetric solutions

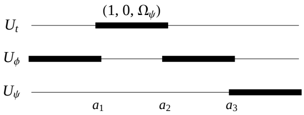
- This rod structure classification can be generalized to the stationary case. The main difference is that the rods acquire non-trivial directions: [Harmark, 2004]



4D Kerr



5D Myers-Perry



rotating black ring

- The rod directions $v_{(k)}$ are the eigenvectors of G evaluated on the rod:

$$G(\rho = 0, z)v_{(k)} = 0 \qquad \text{for} \quad z \in [a_{k-1}, a_k] \, .$$

- For a timelike rod v normalized such that $v^0 = 1$, the rest of the coefficients correspond to the angular velocities of the horizon.

The inverse scattering method [Belinski & Zakharov, 1979]

- The BZ approach consists in replacing the original (non-linear) equation by a system of linear equations (Lax pair).

  One can obtain new solutions by dressing the generating matrix of a known seed solution $G_0(\rho, z)$:

$$G_0 \quad \longrightarrow \quad \Psi_0 \quad \longrightarrow \quad \chi\Psi_0 = \Psi \quad \longrightarrow G$$

◇ $\Psi(\lambda, \rho, z)$ is the generating matrix, required to satisfy $\Psi(0, \rho, z) = G(\rho, z)$.
◇ λ is the spectral parameter, independent of ρ and z.

- Generally, the most challenging step is finding the generating matrix for the seed solution. But for a static seed this is straightforward.

- The dressing matrix χ is chosen to have only simple (real) poles in λ:

$$\chi(\lambda, \rho, z) = 1 + \sum_{k=1}^{n} \frac{R_k(\rho, z)}{\lambda - \widetilde{\mu}_k}, \qquad \text{where} \quad \widetilde{\mu}_k = \begin{cases} \mu_k & \text{for soliton} \\ \overline{\mu}_k & \text{for antisoliton} \end{cases}$$

  This is a solitonic transformation, which adds n new poles to the seed solution.

The inverse scattering method [Belinski & Zakharov, 1979]

The BZ construction is a *purely algebraic* procedure

Starting from a seed matrix G_0, an n-soliton transformation yields a new matrix G:

$$G = G_0 - \sum_{k,l=1}^{n} \frac{(\Gamma^{-1})_{kl}\, G_0 . {m^{(k)}}^T . m^{(l)} . G_0}{\widetilde{\mu}_k \widetilde{\mu}_l} , \qquad \Gamma_{kl} = \frac{m^{(k)} . G_0 . m^{(l)}}{\rho^2 + \widetilde{\mu}_k \widetilde{\mu}_l} ,$$

$$\text{where} \quad m^{(k)} = m_0^{(k)} . \Psi_0^{-1}(\widetilde{\mu}_k, \rho, z) , \quad \text{and} \quad \widetilde{\mu}_k = \pm\sqrt{\rho^2 + (z - a_k)^2} - (z - a_k) .$$

- Input needed: the positions of the solitons a_k and the (constant) BZ vectors $m_0^{(k)}$.
- If the BZ vectors mix the time and spatial Killing directions, then this procedure may yield a rotating version of the original static solution.
- The new conformal factor is simply $e^{2\nu} = e^{2\nu_0} \frac{\det \Gamma}{\det \Gamma_0}$.

The inverse scattering method

- Issue: Generically, after a solitonic transformation the metric no longer obeys $\det G = -\rho^2$.
- Pomeransky proposed an elegant and practical solution for this problem:

  The determinant of the new metric is independent of the BZ vectors.
  Therefore, remove n (anti-)solitons with trivial BZ vectors and then re-add the same (anti-)solitons with more general BZ vectors. [Pomeransky, 2006]

$$\Rightarrow \det G = \det G_0 \, .$$

Conclusion

For static (diagonal) seeds the whole method is algebraic and can be easily implemented in a symbolic manipulation computer program.

Outline

5D minimal supergravity

- 5D minimal supergravity is the simplest supersymmetric extension of 5D vacuum gravity.

- Field content (bosonic sector): — metric $g^{(5)}$, with Ricci scalar $R^{(5)}$;
  — gauge potential $A^{(5)}$, with field strength $F^{(5)} = dA^{(5)}$.

- The Lagrangian has the form of Einstein-Maxwell theory with a Chern-Simons term:

$$\mathcal{L}^{(5)} = R^{(5)} \star 1 - \frac{1}{2} \star F^{(5)} \wedge F^{(5)} - \frac{1}{3\sqrt{3}} F^{(5)} \wedge F^{(5)} \wedge A^{(5)} \,.$$

- Equivalently, the action for the bosonic sector is

$$\mathcal{S}^{(5)} = \frac{1}{16\pi G_5} \int d^5x \left[\sqrt{-g^{(5)}} \left(R^{(5)} - \frac{1}{4} F^{(5)}_{\mu\nu} F^{(5)\,\mu\nu} \right) - \frac{1}{12\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\lambda} F^{(5)}_{\mu\nu} F^{(5)}_{\rho\sigma} A^{(5)}_{\lambda} \right] .$$

where $d^5x = dx^1 \ldots dx^5$. The timelike coordinate is x^5.

Dimensional reduction to 3D

[Mizoguchi & Ohta, 1998]
[Cremmer, Julia, Lü & Pope, 1999]
[Bouchareb et al., 2007]
[Compère, de Buyl, Jamsin & Virmani, 2009]

- Assume existence of two Killing vectors, a spacelike $\partial/\partial x^4$ and a timelike $\partial/\partial x^5$.
- Dimensionally reduce to 3D over the directions x^5 and x^4 using the Kaluza-Klein ansatz:

$$\begin{aligned} ds^2_{(5)} &= e^{\frac{1}{\sqrt{3}}\phi_1+\phi_2} ds^2_{(3)} + e^{\frac{1}{\sqrt{3}}\phi_1-\phi_2}(dx^4+\mathcal{A}^2)^2 - e^{-\frac{2}{\sqrt{3}}\phi_1}(dx^5+\chi_1 dx^4+\mathcal{A}^1)^2 \,, \\ A^{(5)} &= A + \chi_3 dx^4 + \chi_2 dx^5 \,. \end{aligned}$$

- One ends up with 3D fields:

— the 3D metric g;
— two dilatons ϕ_1, ϕ_2;
— three axions χ_1, χ_2, χ_3;
— three one-form potentials $\mathcal{A}^1$, $\mathcal{A}^2$, A.

none of which depend on x^4 and x^5.

Heuristically,

$$g^{(5)}_{\mu\nu} = \left(\begin{array}{c|c|c} g_{ab} & \mathcal{A}^2 & \mathcal{A}^1 \\ \hline \star & \phi_2 & \chi_1 \\ \hline \star & \star & \phi_1 \end{array}\right) .$$

Dualizing gauge potentials into axions

- Remark: The Hodge dual of a one-form potential in 3D is a scalar. Schematically,

$$\star dA_{(1)} = \star F_{(2)} \equiv G_{(1)} = d\chi_{(0)} \,.$$

(Eqs. of motion and Bianchi identities get interchanged.)

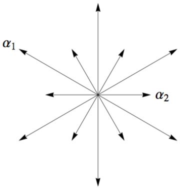
- In order to make the full hidden symmetry apparent one must dualize the one-form potentials into scalars:

$$A \longrightarrow \chi_4 \,, \qquad \mathcal{A}^1 \longrightarrow \chi_5 \,, \qquad \mathcal{A}^2 \longrightarrow \chi_6$$

- We end up with — the 3D metric g; — two dilatons ϕ_1, ϕ_2; — six axions $\chi_1, \ldots, \chi_6$;

and a Lagrangian of the form (schematically):

$$\mathcal{L}^{(3)} = R^{(3)} \star 1 - \frac{1}{2} \star d\vec{\phi} \wedge d\vec{\phi} - \sum_{i=1}^{6} \frac{1}{2} e^{\vec{\alpha}_i \cdot \vec{\phi}} \star d\chi_i \wedge d\chi_i \,.$$



from [Compère et al., 2009]

Each axion χ_i is associated with one of the 6 positive roots $\vec{\alpha}_i$ of the exceptional Lie algebra $\mathfrak{g}_2$.

Non-linear σ-model

1. Write a representative $\mathcal{V}$ for the coset group by exponentiating the Cartan (h_1, h_2) and positive root generators $(e_i, i = 1, \ldots, 6)$ of $\mathfrak{g}_2$ with the dilatons and axions as coefficients:

$$\mathcal{V} = e^{\frac{1}{2}\phi_1 h_1 + \frac{1}{2}\phi_2 h_2}\ e^{\chi_1 e_1}\ e^{-\chi_2 e_2 + \chi_3 e_3}\ e^{\chi_6 e_6}\ e^{\chi_4 e_4 - \chi_5 e_5}\ .$$

This is an upper triangular 7×7 matrix. [Compère, de Buyl, Jamsin & Virmani, 2009]

2. Define a symmetric matrix $M = S^T \mathcal{V}^T \eta \mathcal{V} S$, where S and η are 7×7 matrices of constants.

3. The reduced Lagrangian can be written as

$$\mathcal{L}^{(3)} = R^{(3)} \star 1 - \frac{1}{8}\mathrm{Tr}\left(\star(M^{-1}dM) \wedge (M^{-1}dM)\right)\ .$$

Non-linear σ-model

- The reduced Lagrangian is invariant under global G_2 transformations:

$$\mathcal{V} \longrightarrow k\mathcal{V}g \qquad \Rightarrow \qquad M \longrightarrow M_g = (S^{-1}gS)^T M (S^{-1}gS)\,, \qquad \text{for} \quad g \in G_2\,.$$

(The transformation on $\mathcal{V}$ requires a compensator $k \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ so that it remains upper triangular.)

- Upshot: The reduced Lagrangian is just 3D gravity coupled to a non-linear σ-model for the coset $G_2/\,(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$.

Using the hidden symmetry as a solution generating technique

1. Start with a seed solution of 5D minimal supergravity with two Killing vectors;
2. Reduce solution to 3D gravity coupled to the non-linear σ-model for the resulting 8 scalars;
3. From these 8 scalars construct the 7×7 matrix M;
4. Act on M with an element of G_2 to obtain M_g;
5. Extract from M_g the 8 new scalars;
6. Uplift back to a 5D solution.

Outline

From 3D to 2D [Breitenlohner, Maison & Gibbons, 1988]

- Consider (again) 5D minimal supergravity, and assume existence of 3 commuting Killing vectors, $\partial/\partial x^a$ with $a = 3, 4, 5$.
- Now, take the theory already reduced to 3D and reduce it (along x^3) down to 2D.

  The KK ansatz simplifies considerably because the KK vector B_m can be dropped. [Tomizawa, Yasui & Ishibashi, 2009]

$$g^{(3)}_{\mu\nu} = \begin{pmatrix} \xi^2 \bar{g}^{(2)}_{mn} + \rho^2 B_m B_n & \rho^2 B_m \\ \rho^2 B_n & \rho^2 \end{pmatrix} \quad \longrightarrow \quad g^{(3)}_{\mu\nu} = \begin{pmatrix} \xi^2 \delta_{mn} & 0 \\ 0 & \rho^2 \end{pmatrix}$$

- Taking into account that $\partial M/\partial x^3 = 0$, the 3D Lagrangian (recall)

$$\mathcal{L}^{(3)} = R^{(3)} \star 1 - \frac{1}{8}\mathrm{Tr}\left(\star(M^{-1}dM) \wedge (M^{-1}dM)\right)$$

  descends to

$$\mathcal{L}^{(2)} = 2\xi^{-1} \star d\rho \wedge d\xi - \frac{1}{8}\rho\mathrm{Tr}\left(\star(M^{-1}dM) \wedge (M^{-1}dM)\right) .$$

  Note: now the Hodge dual $\star$ and the differential d are taken over 2D flat space.

From 3D to 2D [Breitenlohner, Maison & Gibbons, 1988]

- The eq. of motion for the field ρ reveals that it is a harmonic function:

$$d \star d\rho = 0 \qquad \Leftrightarrow \qquad \nabla^2 \rho = 0\,.$$

$\Rightarrow$ We can choose ρ to be one of the coordinates on the 2D space, $x^1 = \rho$.
Take the remaining coordinate $x^2 \equiv z$ to be its conjugate harmonic, $dz = - \star d\rho$.

- In coordinates (ρ, z) the equations for the matrix field M can be written as

$$d(\star\rho\, M^{-1} dM) = 0 \qquad \Leftrightarrow \qquad \partial_\rho U + \partial_z V = 0\,.$$

where $U \equiv \rho\, \partial_\rho M\, M^{-1}$ and $V \equiv \rho\, \partial_z M\, M^{-1}$.

- The equations for ξ are

$$\xi^{-1} \partial_\rho \xi = \frac{1}{16\rho} \mathrm{Tr}(U^2 - V^2), \qquad \xi^{-1} \partial_z \xi = \frac{1}{8\rho} \mathrm{Tr}(UV).$$

BZ for 5D minimal SUGRA [Figueras, Jamsin, JVR & Virmani, 2010]

- Observation: the previous equations have the same form as those used as the starting point for the BZ method.

  Indeed, they are exactly equal if one makes the replacements

$$M \longrightarrow G\,, \qquad \xi^2 \longrightarrow \sqrt{\rho}e^{\nu}\,.$$

Upshot

The matrix M which encodes the 8 scalar fields obtained in the reduction from 5D to 3D plays the role of the metric on the Killing fields, G, in the standard BZ construction.

The dualization of 3D vectors into scalars is a crucial step to make the integrability apparent.

The BZ technology for constructing solitonic solutions from known solutions can be applied to 5D minimal supergravity in the presence of $D-2$ commuting Killing vector fields.

- Note: Most likely, the BZ construction can be applied to any theory that reduces to a non-linear σ-model in 3D (these have been classified in [Cremmer, Julia, Lü & Pope, 1999]) and with 'BZ-like' equations of motion when reduced to 2D.

Does BZ preserve the coset? [Figueras, Jamsin, JVR & Virmani, 2010]

- Take a seed matrix $M_0 \in G_2/\,(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$ and dress it through the BZ procedure. Call the resulting matrix M.

  Issue: In general M is *not* guaranteed to be in the coset. ☹

- But in specific cases we can make progress. ☺

  — The vacuum truncation of the matrix M (i.e. $\chi_2 = \chi_3 = \chi_4 = 0$) is block diagonal:

$$M = \begin{pmatrix} M_{\mathrm{SL}(3)}^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & M_{\mathrm{SL}(3)} \end{pmatrix}.$$

  — To preserve this form during the BZ construction, each soliton added to the block $M_{\mathrm{SL}(3)}$ must be accompanied by the addition of an anti-soliton to the block $M_{\mathrm{SL}(3)}^{-1}$.

  — Restrictions can be placed on the BZ vectors so that $M \in G_2/\,(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$.

Combining hidden symmetries and BZ [Figueras, Jamsin, JVR & Virmani, 2010]

- Take a seed M_0 and generate M_1 by a BZ transformation.

  In terms of the generating matrices:

  $$\Psi_1 = \chi\widetilde{\chi}\Psi_0 \equiv \chi\widetilde{\Psi}_0 .$$

$$\begin{tikzcd}
\boxed{M_0} \arrow[r, "g \in G"'] \arrow[d, "\mathrm{BZ}"] & \boxed{M_0^g} \arrow[d, "\mathrm{BZ}_g"] \\
\boxed{M_1} \arrow[r, "g \in G"'] & \boxed{M_1^g}
\end{tikzcd}$$

- Let M_0^g and M_1^g be the matrices generated from M_0 and M_1 by *the same* charging transformation g.

- To generate M_1^g from M_0^g using the ISM we just need to make the replacements

$$m_0^{(k)} \longrightarrow \hat{m}_0^{(k)} = m_0^{(k)}(S^{-1}gS)\,, \qquad \widetilde{\Psi}_0(\lambda,\rho,z) \longrightarrow \widetilde{\Psi}_0^g = (S^{-1}gS)^T\,\widetilde{\Psi}_0\,(S^{-1}gS)\,,$$

Lesson

The ISM is incapable of producing charging transformations. The BZ construction is insensitive to the presence of charges generated by hidden symmetry transformations.

Examples [Figueras, Jamsin, JVR & Virmani, 2010]

1. In the context of vacuum gravity, the 5D Myers-Perry solution has been generated from 5D Schwarzschild using the ISM. This involves a two-soliton transformation. [Pomeransky, 2006]

   We are now able to generate Myers-Perry from Schwarzschild in the context of 5D minimal supergravity. It requires a four-soliton transformation.

2. 5D Reissner-Nordström can be obtained from 5D Schwarzschild by a G_2 transformation.

   Since we know how to combine charging transformations and BZ transformations we can now generate from 5D Reissner-Nordström a charged, doubly rotating black hole in 5D minimal supergravity.

   This exactly reproduces the Cvetič-Youm solution, parametrized by mass $\mathcal{M}$, two angular momenta $\mathcal{J}_\phi, \mathcal{J}_\psi$ and electric charge $\mathcal{Q}$. [Cvetič &Youm, 1996]

$$
\begin{tikzcd}
\boxed{\text{Schwarzschild}} \arrow[r, "k_2 \in K"'] \arrow[d, "\text{BZ}"] & \boxed{\text{Reissner-Nordström}} \arrow[d, "\text{BZ}_{k_2}"] \\
\boxed{\text{Myers-Perry}} \arrow[r, "k_2 \in K"'] & \boxed{\text{Cvetič-Youm}}
\end{tikzcd}
$$

Outline

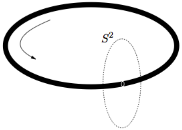
The set-up

- Consider 5D Einstein-Maxwell-dilaton theory, governed by the action

$$S = \frac{1}{16\pi G_N} \int d^5x\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{-a\phi}F_{\mu\nu}F^{\mu\nu}\right) , \qquad \text{with} \quad a = \frac{2\sqrt{2}}{\sqrt{3}} .$$

- The five-dimensional theory naturally supports magnetic one-branes and dipole black rings.

  Can define a local charge by $\mathcal{Q} = \frac{1}{4\pi}\int_{S^2} F$.



from [Emparan, 2004]

- This action can also be obtained from 6D vacuum gravity by performing a Kaluza-Klein reduction on S^1 using the ansatz

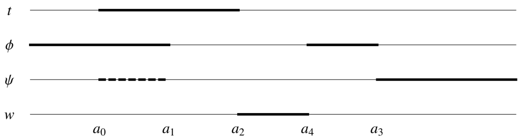
$$ds_6^2 = e^{\frac{\phi}{\sqrt{6}}} ds_5^2 + e^{-\frac{\sqrt{3}\phi}{\sqrt{2}}} (dw + A)^2 .$$

  The sixth dimension is parametrized by w and $F = dA$.

- We want to construct the dipole ring solution of this theory.
  Strategy: Apply the inverse scattering method in 6D and then reduce to 5D.

Seed metric [JVR, Rodriguez & Virmani, 2011]

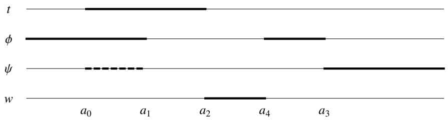
- The seed is taken to be the static (diagonal) metric described by the following rod diagram:



$$ds_6^2 = (G_0)_{ij}\, dx^i dx^j + e^{2\nu_0}(d\rho^2 + dz^2)\,, \qquad G_0 = \mathrm{diag}\left\{-\frac{\mu_0}{\mu_2}, \frac{\rho^2\mu_4}{\mu_1\mu_3}, \frac{\mu_1\mu_3}{\mu_0}, \frac{\mu_2}{\mu_4}\right\}.$$

- This solution is singular and not in itself of direct physical interest.
- The solid rods have positive density $+1/2$ and the dashed rod has negative density $-1/2$.
- The rods add up to an infinite rod with uniform density such that $\det G_0 = -\rho^2$.
- The negative density rod is included in the seed to facilitate adding the S^1 angular momentum to the ring. [Elvang & Figueras, 2007]
- Novel ingredient: the finite rod along the KK direction allows the addition of dipole charge.

Soliton transformations [JVR, Rodriguez & Virmani, 2011]



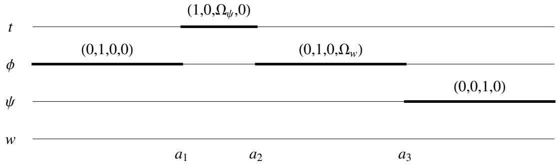
We generate the 6D uplift of the dipole ring solution by a 2-soliton transformation.

1. Perform two 1-soliton transformations on the seed solution G_0, followed by a rescaling:
   — remove an anti-soliton at $z = a_0$ with trivial BZ vector $(1, 0, 0, 0)$;
   — remove a soliton at $z = a_4$ with trivial BZ vector $(0, 0, 0, 1)$;
   — rescale the metric by μ_0/μ_4 (not mandatory) $\longrightarrow$ call it G_0'.
2. Perform now a 2-soliton transformation with G_0' as seed and undo the rescaling:
   — add an anti-soliton at $z = a_0$ with BZ vector $(1, 0, c_1, 0)$;
   — add a soliton at $z = a_4$ with BZ vector $(0, c_2, 0, 1)$;
   — rescale by μ_4/μ_0 $\longrightarrow$ denote final metric by G.
3. Construct $e^{2\nu}$. The result $(G, e^{2\nu})$ is the 6D solution we want.

Appropriately tuning c_1 and c_2 and KK reducing along the w direction we obtain the smooth 5D dipole black ring solution of the theory under consideration.

Dipole black ring uplifted to 6D [JVR, Rodriguez & Virmani, 2011]

We arrive at a metric described by the following rod diagram:



$$\Omega_\psi = \sqrt{\frac{(a_1 - a_0)}{2(a_2 - a_0)(a_3 - a_0)}}\,,$$

$$\Omega_w = \sqrt{\frac{2(a_4 - a_2)(a_4 - a_1)}{(a_3 - a_4)}}\,.$$

- The parameter c_1 must be fixed to avoid a divergence as $z \to a_0$ along the rod $(-\infty, a_0]$. Equally, c_2 must be fixed so that the rod along w merges with the finite rod along ϕ.
- The general solution has a conical deficit, but the balanced solution is regular.
- Parameter counting:

$$\underbrace{\#a_i}_{5} + \underbrace{\#c_j}_{2} - \underbrace{(\text{translational invariance in z})}_{1} - \underbrace{(\text{regularity conditions})}_{2} - \underbrace{(\text{balance condition})}_{1} = 3$$

These 3 parameters encode the mass, one angular momentum and the dipole charge.

Dipole black ring solution [JVR, Rodriguez & Virmani, 2011]

To confirm we have indeed reproduced the dipole black ring of [Emparan, 2004] we have to:

1. Convert from canonical coordinates (ρ, z) to ring coordinates (x, y):

$$\begin{aligned} ds_6^2 &= -\frac{F(y)}{F(x)}\left(dt + C(\nu,\lambda)R\frac{1+y}{F(y)}d\psi\right)^2 + \frac{H(x)}{H(y)}\left(dw + C(\nu,-\mu)R\frac{1+x}{H(x)}d\phi\right)^2 \\ &\quad + \frac{R^2}{(x-y)^2}F(x)H(y)\left[-\frac{G(y)}{F(y)H(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)H(x)}d\phi^2\right]. \end{aligned}$$

   where

$$F(\xi) = 1 + \lambda\xi, \qquad G(\xi) = (1-\xi^2)(1+\nu\xi), \qquad H(\xi) = 1 - \mu\xi,$$

   and

$$C(\nu,\lambda) = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}.$$

2. Perform the dimensional reduction on S^1 down to 5D using the KK ansatz.

We obtain precise agreement with the 5D line element ds_5^2, vector potential A and dilaton ϕ of [Emparan, 2004]. The correct bounds on the parameters are also recovered.

Outline

Conclusions and Outlook

- In part I we have: — extended the well-known BZ method to 5D minimal supergravity.

  — shown how to combine hidden symmetry transformations and BZ transformations in a practical way.

- In part II we have: — described how the ISM in 6D can be used to generate dipole ring solutions of 5D Einstein-Maxwell-dilaton theory with a specific coupling constant.

- Possible extensions: — in principle, the BZ approach can be applied to many more theories that reduce to a non-linear σ-models in 3D.

  — generating more general black rings in the above mentioned 5D Einstein-Maxwell-dilaton theory. (Work in progress.)