Isometric embeddings and asymptotically AdS spacetimes

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Definition (Embedding of differentiable manifold)

A differentiable embedding $\phi : M \to N$ is an injective map (for x, y distinct $\phi(x) \neq \phi(y)$) so that the image $\phi(M)$ is homeomorphic to M and furthermore the induced tangent space map is injective.

Note $dim(N) \geq dim(N)$. In terms of coordinates (σ^a) on M and (x^μ) on N, we write $\phi : \sigma \to x(\sigma)$ and $\phi_* : v^a \to v^a \partial x^\mu / \partial \sigma^a$.

Definition (isometric embedding)

Let (M, h) and (N, g) be (pseudo-)Riemannian manifolds. A smooth embedding $\phi : M \to N$ is isometric if $\phi_* g = h$.

Straightforward examples

- Clifford torus (flat) $S^1 \times S^1 \to \mathbb{E}^4$;
  $(\theta, \phi) \to (a\cos\theta, a\sin\theta, b\cos\phi, b\sin\phi)$.

Sometimes convenient to represent embedded submanifold as set of points satisfying constraint equation(s). e.g.

- Sphere S^n. The image is: $\{X \in \mathbb{E}^{n+1} | X \cdot X = 1\}$.
- Hyperbolic space. $H^n \to \mathbb{M}^{1,n}$. The image is
  $\{X^\mu \in \mathbb{M}^{1,n} | X^0 > 0,\ X^\mu X^\nu \eta_{\mu\nu} = -1\}$.

Straightforward examples continued

Define $\eta_{\mu\nu}^{(2,n-1)} := \mathrm{diag}(-1,+1,...,+1,-1)$.

- Anti de Sitter space. $\widetilde{\mathrm{AdS}}^n \to \mathbb{M}^{2,n-1}$. The image is $\{X^\mu \in \mathbb{M}^{2,n-1} | X^\mu X^\nu \eta_{\mu\nu}^{(2,n-1)} = -1\}$. This has closed timelike curves.

The cosmological AdS space is the universal cover of $\widetilde{\mathrm{AdS}}$. Later we present an embedding of AdS^n into $\mathbb{M}^{2,n}$.

Isometric embedding of the Schwarzschild solution

Embedding found by Kasner (1921) can not be extended beyond the horizon.
An embedding $M \to \mathbb{M}^{5,1}$ was found by [Fronsdal 1959]:

$$X^0 = 2R\sqrt{1-\frac{R}{r}}\cosh(\frac{t}{2R}),$$

$$X^1 = 2R\sqrt{1-\frac{R}{r}}\sinh(\frac{t}{2R}),$$

$$X^2 = \int dr\sqrt{\frac{R}{r}+\frac{R^2}{r^2}+\frac{R^3}{r^3}},$$

and $(X^3)^2 + (X^4)^2 + (X^5)^2 = r^2$. This can be continued across the horizon.
More recent examples see e.g. [Paston, Sheykin 2012].

Theory of isometric embeddings: some history

- Riemann (1850's) introduced manifolds as intrinsically defined objects; modern abstract definition due to Weyl.
- Natural question: are they more general than surfaces in Euclidean space? Schläfli (1873) conjectured that locally an isometric embedding exists in $d = n(n+1)/2$.
- Janet and Cartan (1920's) proved this for the analytic case.
- (Whitney (1936) global embedding theorem in diff. topology.)
- Nash (1956) resolved the global isometric embedding problem for any C^k Riemannian manifold into Euclidean space of large enough dimension.
- Generalised to pseudo-Riemannian manifolds (Clarke 1970, Greene 1970). In particular, any globally hyperbolic spacetime admits G.I.E. into $\mathbb{M}^{d,1}$.
- Many other results

Some applications

- Strings and branes (minimal surfaces)
- Illustrating the geometry and global features of exact solutions of einstein equations.
- Gravity a la string [Regge, Teitelboim]
- Alternative method of studying perturbations/stability

Regarding the last two points, a problem is that there is some degeneracy in the functional derivatives with respect to the embedding coordinate functions. This degeneracy is a result of *isometric bending*. A possible resolution is to consider the function space only in some neighbourhoods of classical solutions which are embedded *freely* in the sense of [Nash 56].

The BTZ black hole ($J = 0$)

In 2+1 dimensional gravity with negative cosmological constant, the solutions satisfy $R^{\mu\nu}{}_{\kappa\lambda} = -\frac{1}{l^2}\delta^{\mu\nu}_{\kappa\lambda}$. We set $l = 1$. The spherically symmetric solution has the static form

$$ds^2 = (r^2 - a^2)d\tau^2 + \frac{dr^2}{r^2 - a^2} + r^2 d\phi^2$$

outside of the event horizon ($r = a$). Kruskal type coordinate system:

$$ds^2 = 4\frac{-dt^2 + dx^2}{(1 + t^2 - x^2)^2} + a^2\frac{(1 - t^2 + x^2)^2}{(1 + t^2 - x^2)^2}d\phi^2 .$$

The domain of the coordinates is $-1 < -t^2 + x^2 < 1$, $\phi \sim \phi + 2\pi$. Singularities at $t^2 - x^2 = 1$, conformal infinity at $x^2 - t^2 = 1$, event horizons $x = \pm t$, bifurcation surface at $x = t = 0$. This covers the maximally extended space-time.

Lemma (S.W. arXiv:1011.3883 gr-qc)

The nonrotating BTZ black hole spacetime can be globally isometrically embedded into $\mathbb{M}^{2,3}$. *The image is* $\{X^\mu \in \mathbb{M}^{2,3} | X^\mu X^\nu \eta^{(2,3)}_{\mu\nu} = -1, (X^1)^2 + (X^2)^2 = \frac{a^2}{1+a^2}(X^0)^2, X^0 > 0\}$ *. The past and future singularities are located at the intersection of the two constraint surfaces with the hyperplane* X^0.

Proof: It can be verified that the following is an embedding

$$X^0(x,t) = \sqrt{1+a^2}\left(\frac{1-t^2+x^2}{1+t^2-x^2}\right),$$

$$X^1(x,t,\phi) = a\left(\frac{1-t^2+x^2}{1+t^2-x^2}\right)\cos\phi\,,\ X^2(x,t,\phi) = a\left(\frac{1-t^2+x^2}{1+t^2-x^2}\right)\sin\phi\,,$$

$$X^3(x,t) = \frac{2x}{1+t^2-x^2},\ X^4(x,y) = \frac{2t}{1+t^2-x^2}.$$

- By lifting the restriction $X^0 > 0$ we obtain two copies of BTZ joined at the singularity, but it is not a true embedding at $X^0 = 0$: the tangent space map is not injective (the BTZ central singularity is a conical singularity).
- From the first constraint equation we conclude that an embedding into $\widetilde{\mathrm{AdS}}_4$ exists.

By the analytic continuation $X^4 \to iX^4$ we obtain:

Lemma

The Euclidean nonrotating BTZ space can be G.I.E. into $\mathbb{M}^{1,4}$. Image is given by constraints $(X^\mu X^\nu \eta_{\mu\nu} = -1,\ (X^1)^2 + (X^2)^2 = \frac{a^2}{1+a^2}(X^0)^2,\ X^0 > 0)$.

This coincides asymptotically with $\{X^\mu \in \mathbb{M}^{1,4} | X^\mu X^\nu \eta_{\mu\nu} = -1, (X^3)^2 + (X^4)^2 = \frac{1}{1+a^2}(X^0)^2, X^0 > 0\}$ which is “thermal AdS", i.e. Euclidean BTZ solution of mass $1/a$ with Euclidean time and ϕ- coordinate reversing roles.

Analytically continuing back to $\mathbb{M}^{2,3}$ by $X^4 \to iX^4$ we find:

Lemma

AdS_3 admits a 1-parameter family of global isometrically embeddings into $\mathbb{M}^{2,3}$. The image is $\{X^\mu \in \mathbb{M}^{2,3} | X^\mu X^\nu \eta^{(2,3)}_{\mu\nu} = -1, (X^3)^2 - (X^4)^2 = \frac{a^2}{1+a^2}(X^0)^2, X^0 > 0\}$.

The isometric bending parameter a is an artefact of the finite temperature of the Euclidean continuation. The above lemma generalises to arbitrary dimension.

Two ball models for hyperbolic space

- **Poincaré Conformal ball.** Unit ball ($y \cdot y < 1$) with metric:

$$g = \frac{4\, dy \cdot dy}{(1 - y \cdot y)^2}$$

  Ideal boundary $y \cdot y = 1$, prototype for conformal infinity.

- **Klein projective ball.** Unit ball ($x \cdot x < 1$) with metric:

$$g = \frac{dx \cdot dx}{1 - x \cdot x} + \frac{(x \cdot dx)^2}{(1 - x \cdot x)^2}$$

  Geodesics, totally geodesic surfaces and geodesic spheres coincide with Euclidean counterparts.

Klein projective model: homogeneous coordinates

We may use homogeneous coordinates (T, X^i). Let C^+ be the interior of the future light cone. Identify points $X^\mu \sim \lambda X^\mu$ for all $\lambda \neq 0$. Introduce (degenerate) metric:

$$g = \frac{dX \circ dX}{(-X \circ X)} + \frac{(X \circ dX)^2}{(X \circ X)^2} = d\left(\frac{X}{\sqrt{X \circ X}}\right) \circ d\left(\frac{X}{\sqrt{X \circ X}}\right) ,$$

- The ball model is obtained by choosing inhomogeneous coordinates $x^i := X^i / T$.
- Ideal boundary is the projective light cone $X \circ X = 0$, $X \sim \lambda X$ i.e. $x \cdot x = 1$. This provides a possible prototype for "projective infinity".

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In terms of the Klein model:

- The Euclidean BTZ is given by the algebraic variety $\{X^\mu | (X^1)^2 + (X^2)^2 = a^2 (X^0)^2/(1+a^2)\}$ in homogeneous coordinates.
- In terms of Klein ball coords: $x^2 + y^2 = a^2/(1+a^2)$. This is a hypercylinder

More generally [Nomizu 73] a hypercylinder over any plane curve in the Klein ball is locally isometric to hyperbolic space. It is fairly straightforward to check that a hypercylinder over any closed plane curve is globally isometric to the nonrotating BTZ! (the conformal metric at infinity is an untwisted torus. The Euclidean BTZ is unique hyperbolic manifold with this conformal boundary.)

Asymptotic conditions

We consider Riemannian manifolds which tend to constant negative curvature at infinity.

- Allow the definition of a generalised ADM total mass and total momentum;
- They are important in the context of semiclassical partition functions and AdS/CFT approach.

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Definition (Conformally compactifiable)

Let (M, g) be a complete noncompact Riemannian manifold. If the we can find:

1. A compact Riemannian manifold $(\hat{M}, \hat{g})$ such that the interior $\hat{M} \setminus \partial\hat{M}$ is diffeomorphic to M;
2. A defining function Ω which has a simple zero at $\partial\hat{M}$ and vanishes nowhere in the interior of $\hat{M}$;
3. A diffeomorphism $\phi : \hat{M} \setminus \partial\hat{M} \to M$ such that $\Omega^2\phi^* g = \hat{g}$,

then we say that M is conformally compactifiable.

Example (The Poincaré ball model revisited)

- $(M, g) = H^n$, hyperbolic space;
- $(\hat{M}, \hat{g})$ is the standard unit Euclidean n-ball;
- $\Omega = (1 - y \cdot y)/2$.

Definition (Locally asymptotically hyperbolic)

If furthermore:

4. the defining function satisfies $\hat{g}^{ab}\hat{\nabla}_a\Omega\hat{\nabla}_b\Omega = \kappa^2$ everywhere on $\partial\hat{M}$

then M is asymptotically locally hyperbolic.

(under Weyl transformation, the Riemann tensor transforms:

$$R^{ab}{}_{cd} = \Omega^2\tilde{R}^{ac}{}_{cd} - 4\Omega\delta^{[a}_{[c}\tilde{\nabla}_{d]}\tilde{\nabla}^{b]}\Omega - 2\delta^{[a}_{[c}\delta^{b]}_{d]}\tilde{\nabla}^e\Omega\tilde{\nabla}_e\Omega\,.$$

Condition 4) $\Rightarrow R^{ab}{}_{cd} \to -2\delta^{[a}_{[c}\delta^{b]}_{d]}\kappa^2$.)

Definition (Conformally compactifiable Lorentzian manifold)

Let (M, g) be a complete noncompact Lorentzian manifold. If the we can find:

1. A compact Lorentzian manifold $(\hat{M}, \hat{g})$, diffeomorphic to $\mathbb{R} \times \Sigma$, where Σ is compact and spacelike, such that the interior $\hat{M} \setminus \partial\hat{M}$ is diffeomorphic to M;
2. A defining function Ω which has a simple zero at $\partial\hat{M}$ and vanishes nowhere in the interior of $\hat{M}$;
3. A diffeomorphism $\phi : \hat{M} \setminus \partial\hat{M} \to M$ such that $\Omega^2 \phi^* g = \hat{g}$,

then we say that M is conformally compactifiable.

Lemma

Let (M_n, g) be some noncompact Riemannian manifold and $P_m = (\mathbb{B}_m, g_P)$ be the poincare ball model for H^m for some $m > n$. If there exists a smooth isometric immersion $\psi : M_n \to \mathbb{B}_m$ such that:

1. *The closure $\overline{\psi(M)}$ is a smooth submanifold with boundary $\partial\psi(M) \subset \partial\mathbb{B}$;*
2. *The tangent space of $\overline{\psi(M)}$ does not coincide with that of $\partial\mathbb{B}$ at the boundary,*

then (M_n, g) is conformally compactifiable.

Proof:

1. Let $\hat{M}_n$ be a compact manifold with boundary and $\varphi : \hat{M}_n \to \bar{\mathbb{B}}_m$ be an immersion into the closed unit ball such that $\varphi(\partial\hat{M}) \subset \partial\mathbb{B}$ and $\varphi(\hat{M} \setminus \partial\hat{M}) = \psi(M)$.
2. We introduce the Poincare metric g_P and standard Euclidean metric g_E on $\mathbb{B}_m$.
3. The function $\Omega \circ \varphi$ will be a valid defining function provided $\psi^* d\Omega \neq 0$ on $\partial\hat{M}$. Since $d\Omega = -y \cdot dy$ this gives condition 2.
4. Then $(\hat{M}, \hat{g} := \varphi^* g_E)$ is a conformal compactification of $(M, g = \psi^* g_P)$ and the diffeomorphism is $\psi^{-1} \circ \varphi$.

The asymptotic form of the physical Riemann tensor is determined by $\hat{g}^{ab}\hat{\nabla}_a\Omega\hat{\nabla}_b\Omega$. This is given in terms of the embedding space coordinates by $\left(\delta^{ij} - \sum_\alpha n^i_{(\alpha)} n^j_{(\alpha)}\right) \partial_i\Omega\partial_j\Omega = y \cdot y - \sum_\alpha (n_{(\alpha)} \cdot y)^2$. Since $n \cdot n = 1$ and on the boundary $y \cdot y = 1$ is satisfied we have the inequalities $0 \leq \sum_\alpha (n_{(\alpha)} \cdot y)^2 \leq 1$. However $\sum_\alpha (n_{(\alpha)} \cdot y)^2 = 0$ is ruled out by condition 2 of the Lemma above.

Lemma: A conformally compactifiable manifold smoothly embedded into the Poincaré ball with smooth behaviour at the boundary is asymptotically hyperbolic with sectional curvatures $-\kappa^2$ if and only if $1 - \sum_\alpha (n_{(\alpha)} \cdot y)^2 = \kappa^2$. Such an embedding is possible only for $\kappa^2 \leq 1$.

When transtated to Klein model we get the simple result:

Lemma

A smooth submanifold of Klein model of hyperbolic space given by homogeneous constraint equations $\Phi^\alpha(X) = 0$ is asymptotically hyperbolic in the conformally compact sense if it is not asymptotically null at the ideal boundary. i.e. $\forall\alpha$ $\partial_\mu\Phi^\alpha\partial_\nu\Phi^\alpha\eta^{\mu\nu} \neq 0$ at $X^\mu X^\nu\eta^{\mu\nu} = 0$.

Outlook

- Classification of embeddings of locally AdS 3-manifolds.
- Free embeddings. Applications to perturbations, Regge-Teitelboim model etc.
- Embedding of asymptotic region. Applications in terms of boundary-bulk holography.