Local Curvature Invariants on Stationary Horizons

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- S. Abdolrahimi and A. A. Shoom,``Geometric properties of static Einstein-Maxwell dilaton horizons with a Liouville potential'', Phys. Rev. D **83**, 104023 (2011) ; arXiv:1103.1171 [hep-th].
- A. A. Shoom, ``Geometric Properties of Stationary and Axisymmetric Killing Horizons'', Phys. Rev. D **91**, 024019 (2015); arXiv:1410.8031 [gr-qc].
- D. N. Page and A. A. Shoom,``Local Invariants Vanishing on Stationary Horizons: A Diagnostic for Locating Black Holes'', Phys. Rev. Lett. **114**, 141102 (2015); arXiv:1501.03510 [gr-qc].

- A Killing horizon is a null hypersurface $\mathcal{H}$ in a pseudo-Riemannian manifold $(\mathcal{M}, g_{\alpha\beta})$ which is invariant with respect to a one-parameter group of the manifold isometries. A generator of the Killing horizon is a null Killing vector $\boldsymbol{\chi}$.

- The null Killing vector $\boldsymbol{\chi}$ has many interesting geometric properties (Carter, Boyer, Wald):

- $\boldsymbol{\chi}$ is the hypersurface orthogonal,

$$\chi_{[\alpha}\nabla_{\beta}\chi_{\gamma]} \mathrel{\mathring{=}} 0\,. \tag{1}$$

- Killing orbits are null geodesics.

- Surface gravity κ,

$$\chi^2_{,\alpha} \mathrel{\mathring{=}} -2\kappa\chi_{\alpha}\,, \qquad \kappa \mathrel{\mathring{=}} const. \tag{2}$$

- If $\kappa \neq 0$ and the Killing orbits lying on $\mathcal{H}$ are complete geodesics, then along each orbit there exists a fixed point where $\chi^{\alpha} = 0$. A set of such points forms a spacelike surface of bifurcation of $\mathcal{H}$.

- Due to special features of a Killing horizon, the spacetime structure takes a special form on and in the vicinity of it.

- Symmetry enhancement, reducibility of Killing tensor, and simplification of the Einstein equations for an extremal Killing horizon $\kappa \mathrel{\hat{=}} 0$ (Kunduri, Lucietti, Reall; Bardeen & Horowitz).

- Vacuum and electrovac space-times of local 4D Weyl-type black holes of Petrov type I become of Petrov type D on the Killing horizon: "appearance" of two repeated principal null directions on $\mathcal{H}$ (Papadopoulos & Xanthopoulos; Abdolarhimi, Frolov, & Shoom).

- Space-time scalar curvature invariants get greatly simplified on a Killing horizon.

- Regular Killing horizons: finite curvature invariants.

- Example: Kretschmann scalar $\mathcal{K} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ on a Killing horizon of 4D electrovacuum static space-time

$$\mathcal{K} \mathrel{\hat{=}} 3\left(\mathcal{R} + F^2\right)^2 + 2F^4\,.$$

- Stationary and axisymmetric Killing horizons: space-time split.
- Metric of a 4D stationary and axisymmetric space-time in (1+1+2)-form.

$$
\begin{aligned}
ds^2 &= (\epsilon k^2 + \omega^c\omega_c)dt^2 + 2\omega_a dt dx^a + \gamma_{ab}dx^a dx^b \,, \\
\gamma_{ab}dx^a dx^b &= -\epsilon\kappa^{-2}dk^2 + h_{AB}dx^A dx^B \,, \\
h_{AB}dx^A dx^B &= h_{xx}dx^2 + h_{\phi\phi}d\phi^2 \,, \qquad \epsilon = \pm 1 \,, \qquad \omega^a = \omega\delta^a{}_\phi \,.
\end{aligned}
$$

- The metric $g_{\alpha\beta}$ has a two-parameter abelian group of isometries $\{\varphi_t, \varphi_\phi\}$.
- The generators of the group are the commuting nonorthogonal Killing vector fields $\boldsymbol{\xi}_{(t)}$ and $\boldsymbol{\xi}_{(\phi)}$: $\xi^\alpha_{(t)} = \delta^\alpha_t$, $\xi^{[\alpha}_{(t)}\nabla^\beta\xi^{\gamma]}_{(t)} \neq 0$ and $\xi^\alpha_{(\phi)} = \delta^\alpha_\phi$.
- Killing vector field: $\boldsymbol{\chi} = \boldsymbol{\xi}_{(t)} + \Omega\boldsymbol{\xi}_{(\phi)}$, $\qquad \omega = \frac{g_{t\phi}}{g_{\phi\phi}} \mathrel{\hat=} -\Omega = const.$
- Killing horizon: $k = 0$,

$$
\boldsymbol{\chi}\cdot\boldsymbol{\chi} = \epsilon k^2 + (\omega^2 + 2\omega\Omega + \Omega^2)\gamma_{\phi\phi} \mathrel{\hat=} 0 \,.
$$

- The Einstein equations:

$$R^\alpha_\beta = \Lambda\delta^\alpha_\beta + 8\pi(T^\alpha_\beta - \frac{1}{2}T\delta^\alpha_\beta)\,, \qquad T = T^\alpha_\alpha\,. \tag{1}$$

- The electromagnetic stress-energy tensor:

$$T^\alpha_\beta = \frac{1}{4\pi}(F^{\alpha\gamma}F_{\beta\gamma} - \frac{1}{4}\delta^\alpha{}_\beta F^2)\,, \qquad T = 0\,, \qquad F^2 = F_{\alpha\beta}F^{\alpha\beta}\,. \tag{2}$$

- The 4-vector potential $\boldsymbol{A}$:

$$\mathcal{L}_{\boldsymbol{\xi}_{(t)}}\boldsymbol{A} = 0\,, \qquad \mathcal{L}_{\boldsymbol{\xi}_{(\phi)}}\boldsymbol{A} = 0\,, \qquad A_{[\alpha}\xi_{(t)\beta}\xi_{(\phi)\gamma]} = 0\,. \tag{3}$$

- As a result, $\boldsymbol{A}$ depends only on the k and x coordinates and can be presented in the form

$$A_\alpha = -\Phi\delta_\alpha{}^t + \mathcal{A}\delta_\alpha{}^\phi\,. \tag{4}$$

- Curvature invariants on the Killing horizon:

$$\mathcal{K}_1 = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \mathrel{\hat{=}} 3\left(\mathcal{R} + \epsilon\tilde{\mathcal{E}} - \frac{2}{3}\Lambda\right)^2 + 4\epsilon H_{abc}H^{abc} + 2\tilde{\mathcal{E}}^2 + \frac{8}{3}\Lambda^2\,,$$

$$\mathcal{K}_2 = {}^*R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \mathrel{\hat{=}} 6\epsilon^{aAB}H_{aAB}\left(\mathcal{R} + \epsilon\tilde{\mathcal{E}} - \frac{2}{3}\Lambda\right)\,,$$

$$\mathcal{K}_3 = {}^*R^*_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \mathrel{\hat{=}} -3\left(\mathcal{R} + \epsilon\tilde{\mathcal{E}} - \frac{2}{3}\Lambda\right)^2 - 4\epsilon H_{abc}H^{abc} + 2\tilde{\mathcal{E}}^2 - \frac{8}{3}\Lambda^2\,,$$

$$R_{\alpha\beta}R^{\alpha\beta} \mathrel{\hat{=}} \tilde{\mathcal{E}}^2 + 4\Lambda^2\,, \qquad R = 4\Lambda\,.$$

Here

$$H_{abc} = 2\epsilon\bar{\mathcal{S}}_{a[b|c]}\,, \qquad \bar{\mathcal{S}}_{ab} = -k^{-1}\omega_{(a|b)}\,, \qquad \tilde{\mathcal{E}} = 16\pi\mathcal{E}\,.$$

How may one locate a black hole?

The location of a black hole is a delicate issue!

- Generally, the location of the event horizon depends on the future evolution of the space-time.
- However, for stationary black holes one may look for a local invariant which is generically nonzero off the horizon but vanishes on the horizon.

- Scalar polynomial curvature invariant.
- Example 1: Schwazschild space-time,

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2)\,, \;\; f = 1 - \frac{2M}{r}\,.$$

Karlhede invariant :

$$I_K \equiv R^{\alpha\beta\gamma\delta;\epsilon} R_{\alpha\beta\gamma\delta;\epsilon} = \frac{720 M^2 (r - 2M)}{r^9}\,.$$

A. Karlhede, U. Lindström, and J. E. Aman, Gen. Rel. Grav. **14**, 569 (1982)

- Example 2: Kerr space-time,

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2$$

$$+ \Sigma\, d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2\,,$$

$$\Sigma = r^2 + a^2\cos^2\theta\,,\ \ \Delta = r^2 - 2Mr + a^2$$

- Horizons:

$$r_\pm = M \pm \sqrt{M^2 - a^2}$$

- Abdelqader-Lake invariant:

$$Q_2 = \frac{a^2 \sin^2\theta (r^2 - 2Mr + a^2)}{M^2 (r^2 + a^2 \cos^2\theta)}$$

Majd Abdelqader and Kayll Lake, Phys. Rev. D **114**, 141102 (2015)

- Curvature invariants:

$$I_1 \equiv C_{\alpha\beta\gamma\delta}\, C^{\alpha\beta\gamma\delta}\,, \;\; I_2 \equiv C^*{}_{\alpha\beta\gamma\delta}\, C^{\alpha\beta\gamma\delta}\,,$$

$$I_3 \equiv \nabla_\mu C_{\alpha\beta\gamma\delta}\, \nabla^\mu C^{\alpha\beta\gamma\delta}\,, \;\; I_4 \equiv \nabla_\mu C_{\alpha\beta\gamma\delta}\, \nabla^\mu C^{*\,\alpha\beta\gamma\delta}\,,$$

$$I_5 \equiv k_\mu k^\mu\,, \;\; I_6 \equiv l_\mu l^\mu\,, \; I_7 \equiv k_\mu l^\mu$$

- Here

$$k_\mu \equiv -\nabla_\mu I_1\,, \;\; l_\mu \equiv -\nabla_\mu I_2$$

- Abdelqader-Lake invariant:

$$Q_2 = \frac{(I_5 + I_6)^2 - (12/5)^2 \left({I_1}^2 + {I_2}^2\right) \left({I_3}^2 + {I_4}^2\right)}{108 \left({I_1}^2 + {I_2}^2\right)^{5/2}}$$

- Complex syzygy:

$$\nabla_\mu (I_1 + iI_2) \nabla^\mu (I_1 + iI_2) = \frac{12}{5} (I_1 + iI_2) (I_3 + iI_4)$$

- Real syzygies:

$$I_6 - I_5 + \frac{12}{5} (I_1 I_3 - I_2 I_4) = 0$$

$$I_7 = \frac{6}{5} (I_1 I_4 + I_2 I_3)$$

- Abdelqader-Lake invariant:

$$27\left({I_1}^2 + {I_2}^2\right)^{5/2} Q_2 = 2\,\|dI_1 \wedge dI_2\|^2\,,$$

$$\|dI_1 \wedge dI_2\|^2 = \tfrac{1}{2}[(k_\mu k^\mu)(l_\nu l^\nu) - (k_\mu l^\mu)(l_\nu k^\nu)]$$

- Another syzygy:

$$\Im\left[(I_1 + iI_2)^4\,(I_3 - iI_4)^3\right] = 0$$

- Theorem:

For a spacetime of local cohomogeneity n that contains a stationary horizon (a null hypersurface that is orthogonal to a Killing vector field that is null there and hence lies within the hypersurface and its null generator) and which has n scalar polynomial curvature invariants $S^{(i)}$ whose gradients are well-defined there, the n-form wedge product $W = dS^{(1)} \wedge dS^{(2)} \wedge \cdots \wedge dS^{(n)}$ has zero squared norm on the horizon,

$$\|W\|^2 \equiv \frac{1}{n!} \delta^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n} g^{\beta_1 \gamma_1} \cdots g^{\beta_n \gamma_n} S^{(1)}_{;\alpha_1} \cdots S^{(n)}_{;\alpha_n} S^{(1)}_{;\gamma_1} \cdots S^{(n)}_{;\gamma_n} \mathrel{\hat{=}} 0 \,.$$

- Proof:

  - $D-$ dimensional stationary spacetime.
  - $m-$ maximal dimension of the orbits of the local isometry group.
  - $n = D - m$ is the local cohomogeneity of the spacetime.
  - $\xi^\mu-$ Killing vector field, which is the generator of the spacetime horizon and which is orthogonal to the null horizon hypersurface. ξ^μ is timelike in a neighbourhood outside the horizon.

- Let $\{S^{(i)}, i = 1, ..., n\}$ be a set of functionally independent, nonconstant scalar polynomial curvature invariants.

- Let $\{dS^{(i)}, i = 1, ..., n\}$ be the set of exterior derivatives with components $S^{(i)}_{;\mu} = \nabla_\mu S^{(i)}$.

- The gradients lie within the $n-$ dimensional local cohomogeneity part of the cotangent space.

- The $n-$ form $W = dS^{(1)} \wedge dS^{(2)} \wedge \cdots \wedge dS^{(n)}$ is proportional to the volume form of the $n-$ dimensional part of the cotangent space.

- The proportionality depends on the spacetime point and is nonzero at the points where the gradients $dS^{(i)}$ are linearly independent.

- The hodge dual $*W$ is an $m-$ form, which is proportional to the volume form of the $m-$ dimensional part of the cotangent space generated by the local isometries.

- On the horizon $*W$ is null, thus
$\| * W\|^2 = \|W\|^2 \mathrel{\hat{=}} 0.$

- Remarks:

- The n scalar polynomial curvature invariants $S^{(i)}$ should be chosen to be functionally independent so that $\|W\|^2$ is positive at generic points in the space-time outside the horizon.

- For a spacetime of local cohomogeneity n with a Killing horizon that is a locally unique hypersurface, are there always enough functionally independent scalar polynomial curvature invariants so that one can choose n of them such that generically $\|W\|^2 \neq 0$?

- Based upon the works

A. Coley, S. Hervik, and N. Pelavas, Class. Quantum Grav. **26**, 025013 (2009)

S. Hervik and A. Coley, Class. Quantum Grav. **27**, 095014 (2010)

A. Coley, S. Hervik, and N. Pelavas, Class. Quantum Grav. **27**, 102001 (2010)

the answer is rather 'yes'.
However, a rigorous proof is needed.

Examples: (D=4)

- Spherically symmetric static black hole: $m = 3, n = 1$. $S = S(r)$, $g^{\alpha\beta} S_{;\alpha} S_{;\beta} \mathrel{\mathring{=}} 0$.
- Stationary axisymmetric black hole: $m = 2, n = 2$. I_1, I_2, $\|dI_1 \wedge dI_2\|^2 \mathrel{\mathring{=}} 0$.
- Distorted static black hole: $m = 1, n = 3$. I_1, I_3, I_5, $\|dI_1 \wedge dI_3 \wedge dI_5\|^2 \mathrel{\mathring{=}} 0$.
- Kerr-NUT-(A)dS black hole: $m = 2, n = 2$. $S^{(1)} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}, S^{(2)} = R^{\alpha\beta}{}_{\gamma\delta} R^{\gamma\delta}{}_{\epsilon\zeta} R^{\epsilon\zeta}{}_{\alpha\beta}$. Generically, away from the horizon and axes $\|dS^{(1)} \wedge dS^{(2)}\|^2 \neq 0$, $\|dS^{(1)} \wedge dS^{(2)}\|^2 \mathrel{\mathring{=}} 0$.

However, there are also certain hypersurfaces away from the horizons and axes on which the wedge product vanishes!

Even if we take the three scalar polynomial curvature invariants $S^{(1)} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, $S^{(2)} = R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\delta}{}_{\epsilon\zeta}R^{\epsilon\zeta}{}_{\alpha\beta}$, and $S^{(3)} = R_{\alpha\beta\gamma\delta;\epsilon}R^{\alpha\beta\gamma\delta;\epsilon}$ and construct the three pairs of wedge products, $W_{12} = dS^{(1)} \wedge dS^{(2)}$, $W_{23} = dS^{(2)} \wedge dS^{(3)}$, and $W_{31} = dS^{(3)} \wedge dS^{(1)}$, each of them will vanish on certain hypersurfaces. Moreover, the sum of the three squared norms of the wedge products vanishes at certain sets of points in the spacetime away from the stationary horizons and fixed points of the isometries.

- General procedure:

Let us take $\{S^{(i)}, i = 1, n + p\}$ with sufficiently large p. Then there are $N = (n + p)!/(n!p!)$ ways to form the $n-$form

$$W_{i_1 \cdots i_n} = dS^{(i_1)} \wedge \cdots \wedge dS^{(i_n)} \,.$$

One can multiply each squared norm $\|W_{i_1 \cdots i_n}\|^2$ by

$$P_{i_1 \cdots i_n} = \|dS^{(j_1)}\|^2 \times \cdots \times \|dS^{(j_p)}\|^2 \,, j_k \neq i_l \,; k, l = 1, n+p$$

and then sum over the N choices to get an invariant

$$I_{(p)} = \sum_{i_1 <, \ldots, < i_n \geq 1}^{N} P_{i_1 \cdots i_n} \|W_{i_1 \cdots i_n}\|^2 \,.$$

Summary

- We constructed local invariants on any stationary horizon including vanishing ones.
- The construction of the vanishing invariants doesn't assume the Einstein equations.
- The invariants might be useful for numerically estimating the location of horizons.
- Nonstationary spacetimes: defining curvature invariant quasi-horizons.

Thank you!