TÉCNICO LISBOA

CENTRA

Static Electrically Charged Shells: Normal Shells and Tension Shells

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VIII Black Holes Workshop | 21 – 22 December
Lisboa, Portugal

Plan

- Analyze the junction of an interior Minkowski spacetime with an exterior Reissner-Nordström spacetime separated by a static perfect fluid thin shell.

  - Study the properties of a thin shell placed at any allowed sub-region of the maximally extended Reissner-Nordström spacetime.

  - The cases of pressure and tension shells arise naturally depending on the sub-region where the shell is considered.

  - Analyze the energy conditions verified by the thin shell for each case.

Israel Junction Formalism

- Consider two spacetimes $(\mathcal{V}^-, g^-)$ and $(\mathcal{V}^+, g^+)$ matched at a surface Σ, forming a new spacetime $(\mathcal{V}, g)$. The new spacetime $(\mathcal{V}, g)$ is a valid solution of the Einstein field equations if:

  - $h^{\pm}_{ij} \equiv g^{\pm}_{\alpha\beta} e^{\alpha}_i e^{\beta}_j$, where e_a are the tangent vectors to Σ, are such that
  $$[h_{ab}]_{\pm} \equiv h^{+}_{ab} - h^{-}_{ab} = 0;$$

  - there is a jump on the extrinsic curvature $K_{ij} \equiv \nabla_{\alpha} n_{\beta} e^{\alpha}_i e^{\beta}_j$, where n^{α} is the normal to Σ, then a thin shell with energy momentum tensor
  $$S_{ab} = -\frac{\varepsilon}{8\pi}\left([K_{ab}]_{\pm} - h_{ab}[K]_{\pm}\right),$$
  where $n_{\alpha} n^{\alpha} = \varepsilon$ and $K = h^{ab} K_{ab}$, is present at Σ.

Interior Minkowski Spacetime

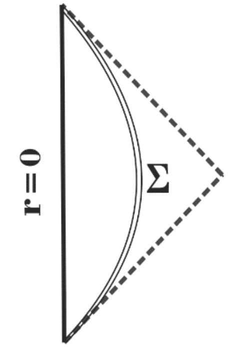
- Consider the interior Minkowski spacetime with metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

- A static shell in the Minkowski spacetime must be time-like.

- The radial coordinate of the shell's, as seen from $\mathcal{V}^-$, is described by a function $R(\tau)$, such that

$$\frac{dR}{d\tau} = 0$$



- Using the definition of the extrinsic curvature:

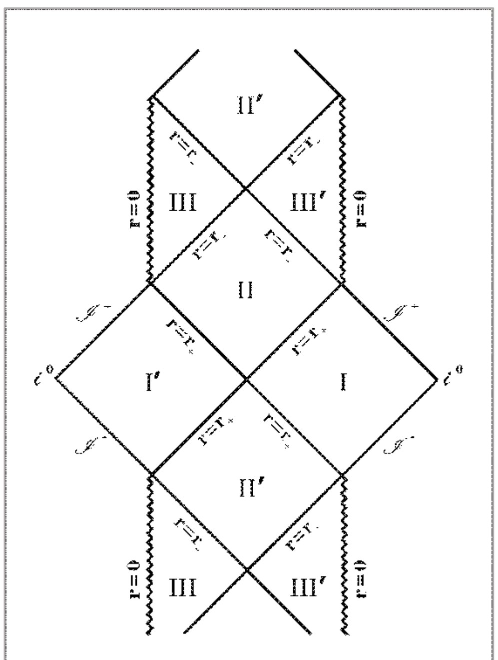
$$K^{\tau}_{-\tau} = 0\,,$$
$$K^{\theta}_{-\theta} = \frac{1}{R}\,,$$
$$K^{\varphi}_{-\varphi} = \frac{1}{R}\,.$$

5. Exterior Reissner-Nordström Spacetime

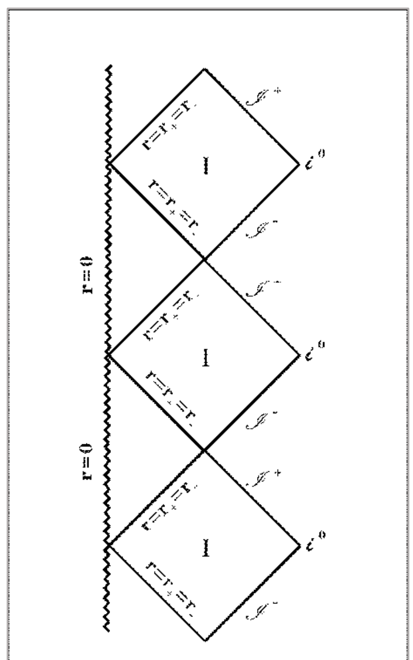
- The Reissner-Nordström solution describes three distinct spacetimes depending on the charge to mass ratio.

$$ds^2 = -\phi(r)dt^2 + \phi^{-1}(r)dr^2 + r^2 d\Omega^2,$$

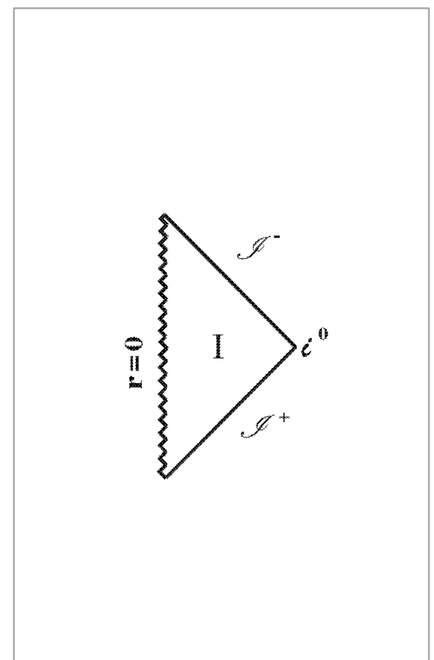
$$\phi(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$



Non extremal Reissner-Nordström spacetime



Extremal Reissner-Nordström spacetime



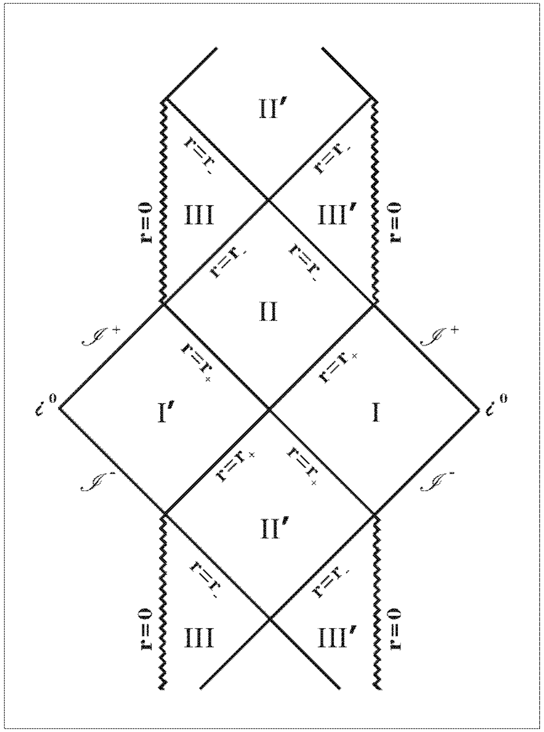
Overcharged Reissner-Nordström spacetime

Non Extremal Reissner-Nordström Spacetime

➡ Start by studying the non-extremal case.

➡ We shall allow the shell to be placed at any sub-region.

- It is convenient to use Kruskal-Szekeres coordinates.
- Although it is possible to define a coordinate system that covers the entire RN spacetime the metric becomes too complicated.



- Work instead with two coordinate patches each well behaved in a neighborhood of the event horizon at $r = r_+$ or in a neighborhood of the Cauchy horizon at $r = r_-$.

Non Extremal Reissner-Nordström Spacetime

- Consider the coordinate patch without coordinate singularity at $r = r_+$.

- Introducing the coordinates (T, X, θ, φ) such that the RN metric is given in the new coordinate system by

$$ds^2 = \frac{16m^2}{r^2} R_+^2 e^{-\frac{r}{R_+}} \left(\frac{r - r_-}{2m}\right)^{1+\left(r_-/r_+\right)^2} \left(dX^2 - dT^2\right) + r^2 (X, T)\, d\Omega^2$$

where

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

$$R_\pm = \frac{r_\pm^2}{r_+ - r_-}$$

$$X^2 - T^2 = e^{\frac{r}{R_+}} \left(\frac{r - r_+}{2m}\right) \left(\frac{r - r_-}{2m}\right)^{-\left(r_-/r_+\right)^2}$$

Non Extremal Reissner-Nordström Spacetime

- From the 1st junction condition the radial coordinate of the shell as seen from $\mathcal{V}^+$ is the same as seen from $\mathcal{V}^-$, such that

$$ds^2\Big|_{\Sigma} = -d\tau^2 + R^2 d\Omega^2$$

- Since the shell is assumed static

$$X^2 - T^2 = \text{constant}$$

- From the above relation

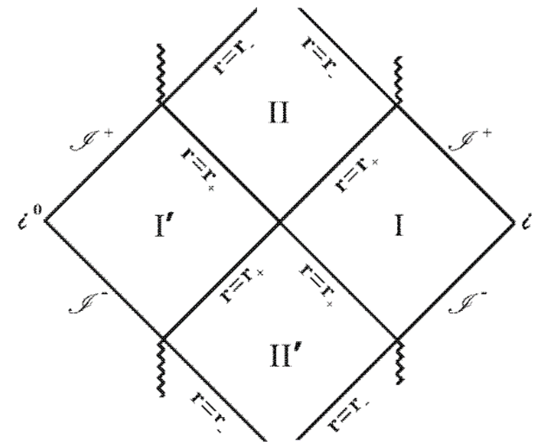
$$\frac{\partial X}{\partial \tau} = \frac{T}{X}\frac{\partial T}{\partial \tau}$$

Non Extremal Reissner-Nordström Spacetime

- The 4-velocity of an observer co-moving with the shell is

$$U_{+}^{\alpha} = \sqrt{\frac{g^{XX}}{X^2 - T^2}}\,(X, T, 0, 0)$$

- Physically, the above expression is only valid if $X^2 - T^2 > 0$.
  - The shell must be either in sub-region I or I′.
  - Expected, since it is not possible to have a static time-like shell at the black hole region.



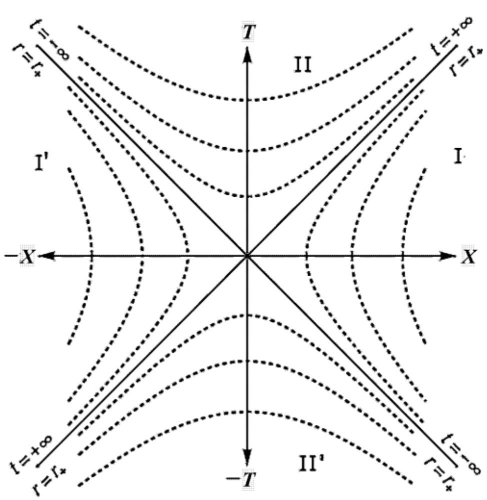
Non Extremal Reissner-Nordström Spacetime

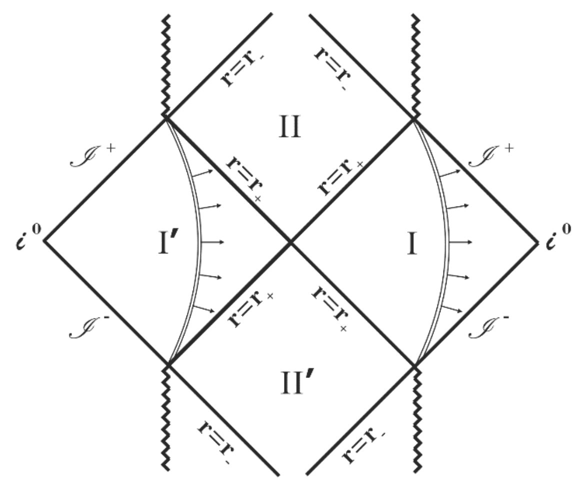
- From the orthogonality and normalization equations the expression for the normal to the hypersurface Σ is

$$n_{+\alpha} = \pm\sqrt{\frac{g_{XX}}{X^2 - T^2}}\,(-T, X, 0, 0)$$

- Relate the sign, therefore the direction, of the normal with the sub-region where the shell is placed.

$$n_{+\alpha} = \mathrm{sgn}\,(X)\,\sqrt{\frac{g_{XX}}{X^2 - T^2}}\,(-T, X, 0, 0)$$





Non Extremal Reissner-Nordström Spacetime

- Substituting the expressions for components of the normal, the non-null components of the extrinsic curvature of Σ when seen by $\mathcal{V}^+$ are

$$K^{\tau}_{+\tau} = \frac{\operatorname{sgn}(X)}{2R^2 A}\left(r_+ + r_- - 2\frac{r_- r_+}{R}\right)$$

$$K^{\theta}_{+\theta} = K^{\varphi}_{+\varphi} = \frac{\operatorname{sgn}(X)}{R}A$$

with

$$A = \sqrt{\frac{(R - r_+)(R - r_-)}{R^2}}$$

Non Extremal Reissner-Nordström Spacetime

- Considering the shell is composed by a perfect fluid

$$S^{ab} = \sigma U^a U^b + p\left(h^{ab} - U^a U^b\right)$$

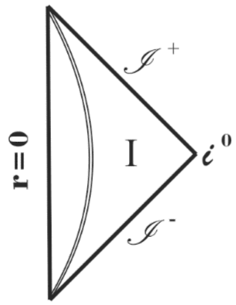
- Comparing with the 2nd junction condition

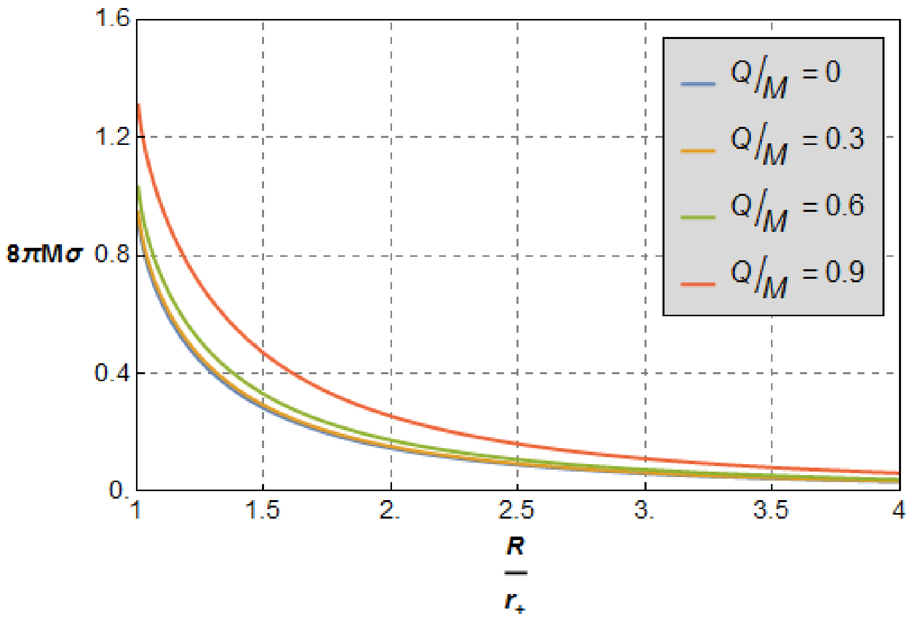
$$8\pi\sigma = \frac{2}{R}\left(1 - \text{sgn}\left(X\right)A\right) ,$$

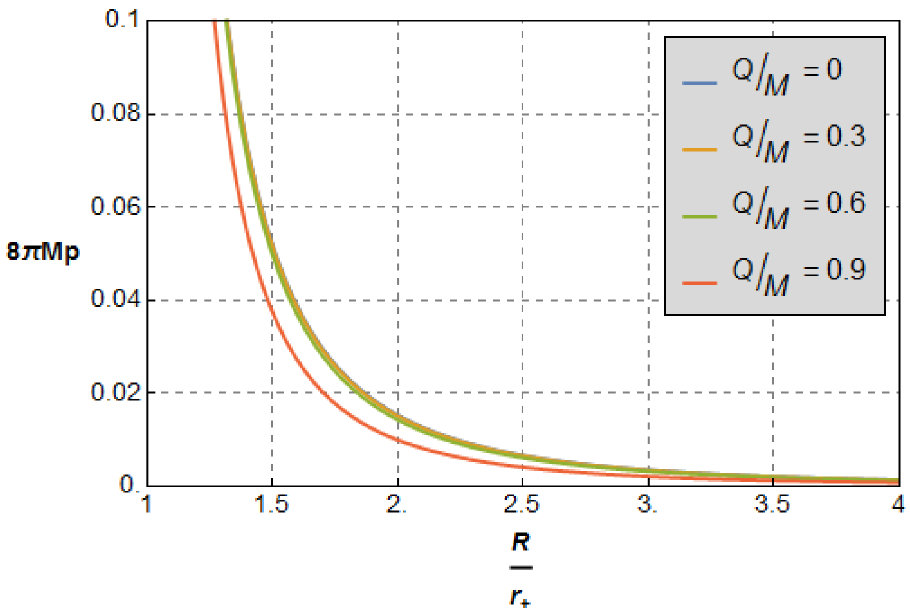
$$8\pi p = \frac{\text{sgn}\left(X\right)}{2RA}\left[\left(1 - \text{sgn}\left(X\right)A\right)^2 - \frac{r_- r_+}{R^2}\right]$$

Non Extremal Reissner-Nordström Spacetime

- Energy density and pressure support of a thin shell at region I

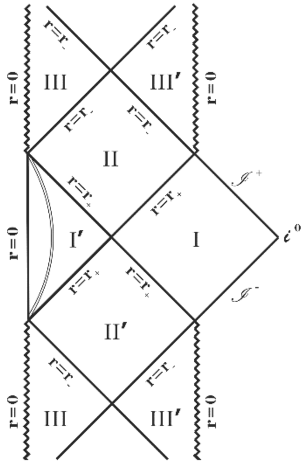


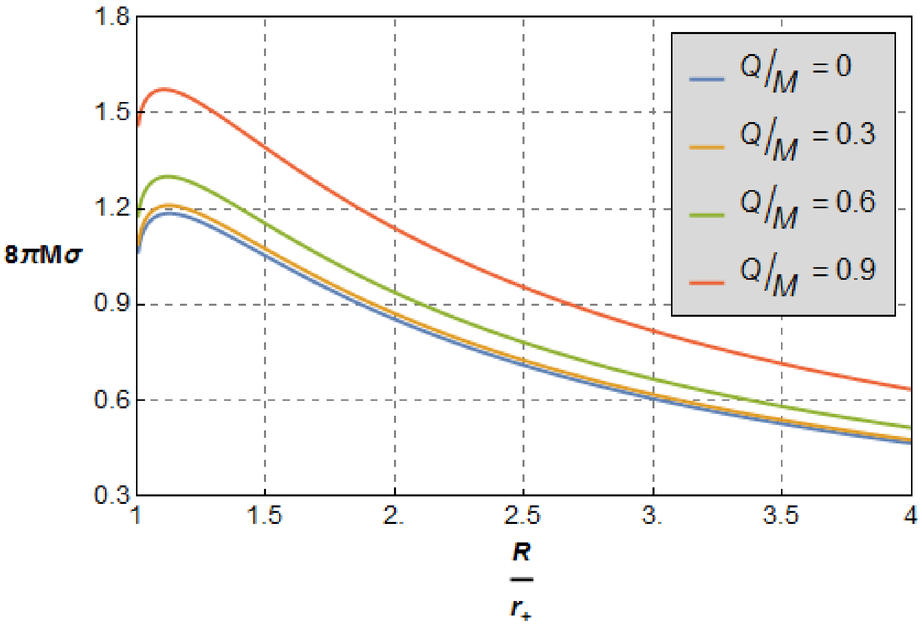


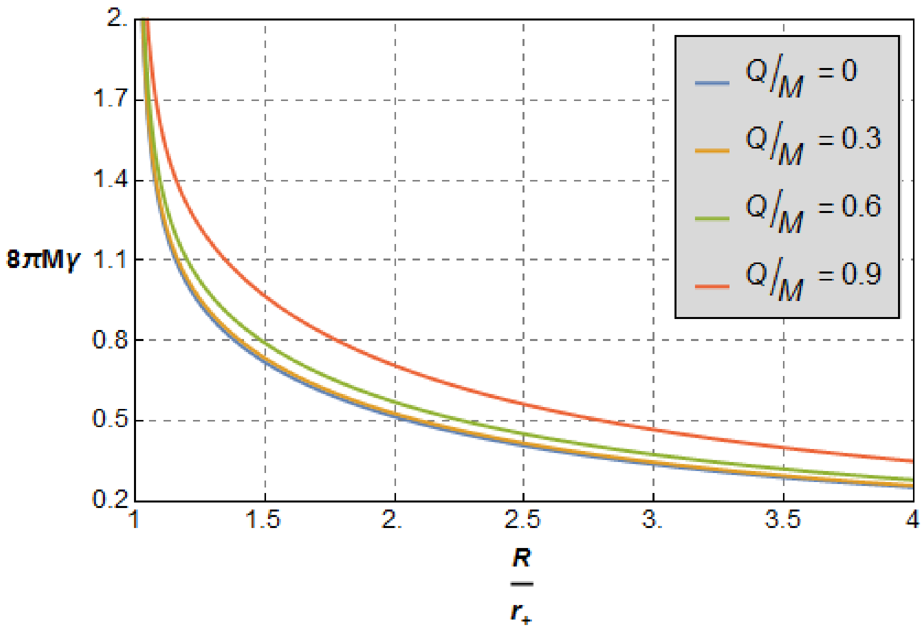


Non Extremal Reissner-Nordström Spacetime

- Energy density and tension support of a thin shell at region I'







Exterior Reissner-Nordström Spacetime

$$8\pi\sigma = \frac{2}{R}\left(1 - \mathrm{sgn}\left(X\right)A\right) \ ,$$

$$8\pi p = \frac{\mathrm{sgn}\left(X\right)}{2RA}\left[\left(1 - \mathrm{sgn}\left(X\right)A\right)^2 - \frac{r_- r_+}{R^2}\right]$$

- The equations for the properties of the thin shell only depend on the Kruskal-Szekeres coordinates in the $\mathrm{sgn}(X)$ term.
- The radial coordinate R is well defined for every sub-region of the non-extremal, extremal and overcharged Reissner-Nordström spacetime.

Exterior Reissner-Nordström Spacetime

- Introducing a new parameter ξ such that
  - $\xi = +1$, if the normal points in the direction of increasing radial coordinate.
  - $\xi = -1$, if the normal points in the direction of decreasing radial coordinate.

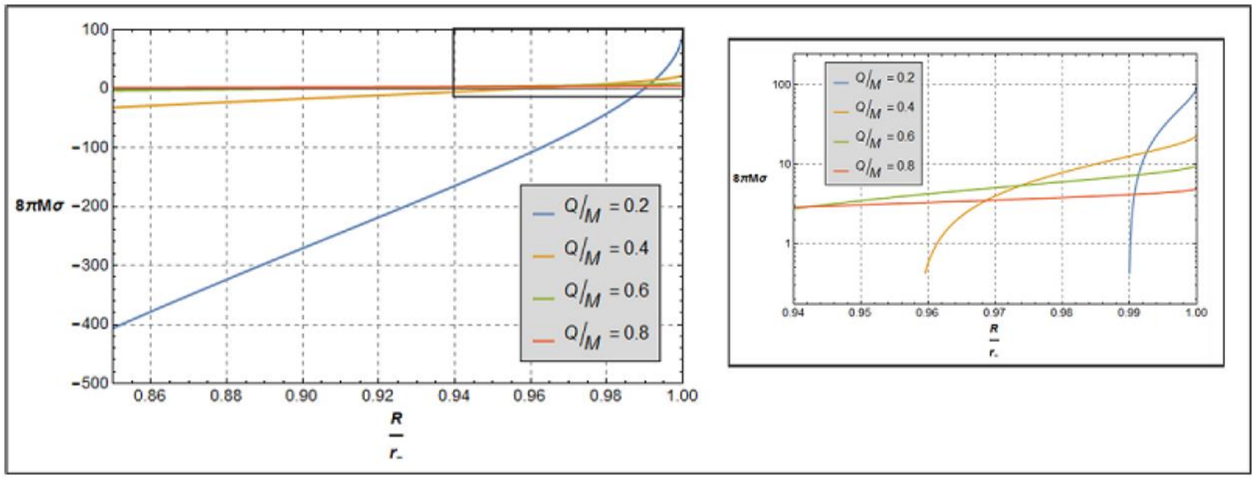
$$8\pi\sigma = \frac{2}{R}\left(1-\xi A\right) ,$$

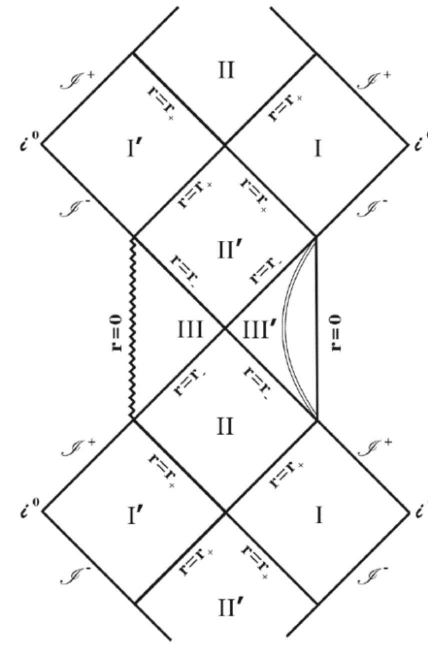
$$8\pi p = \frac{\xi}{2RA}\left[\left(1-\xi A\right)^2 - \frac{r_- r_+}{R^2}\right]$$

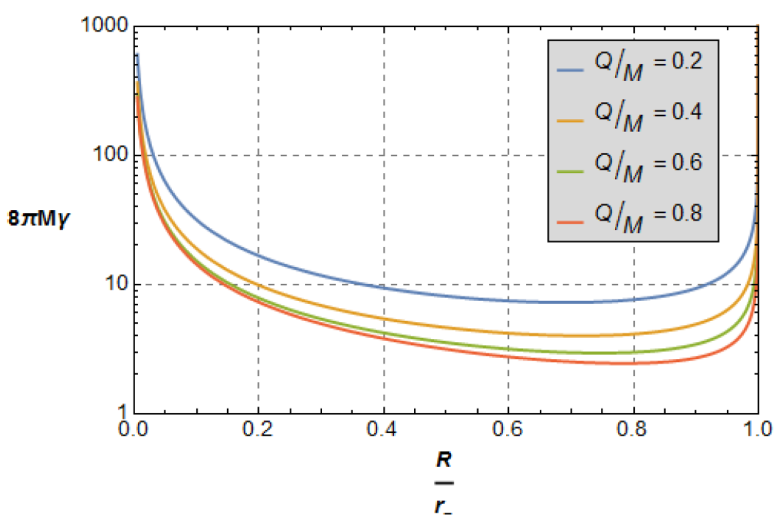
- The properties of the thin shell only depend on the radial coordinate of the shell, R, hence well defined for the non-extremal, extremal and overcharged external spacetime.

Non Extremal Reissner-Nordström Spacetime

- Properties of a thin shell at region III’ of a non extremal Reissner-Nordström spacetime

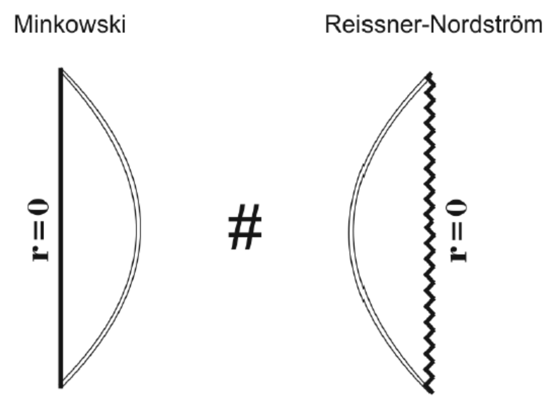


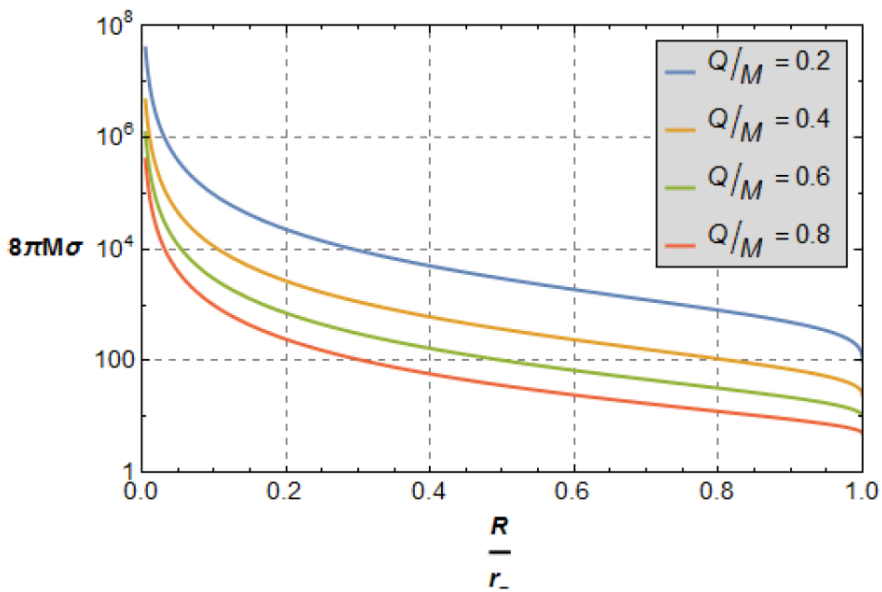


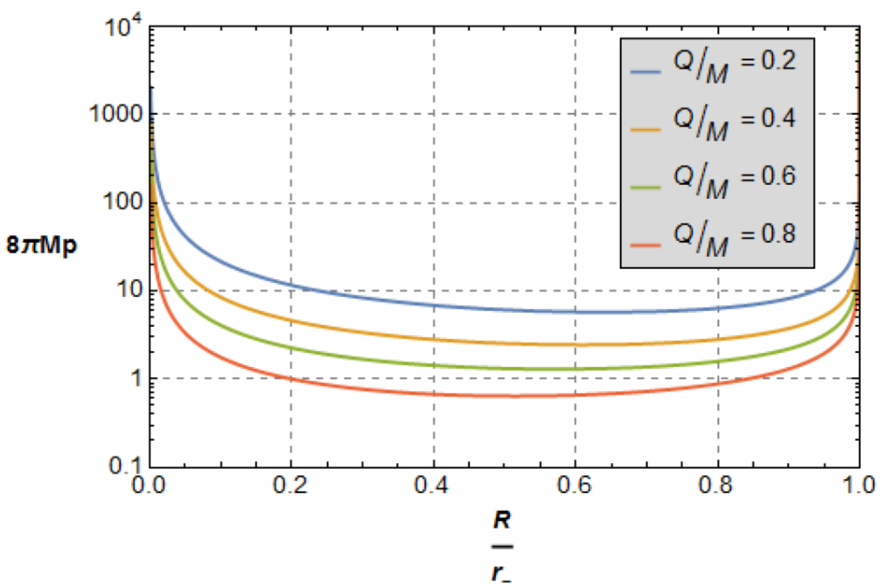


Non Extremal Reissner-Nordström Spacetime

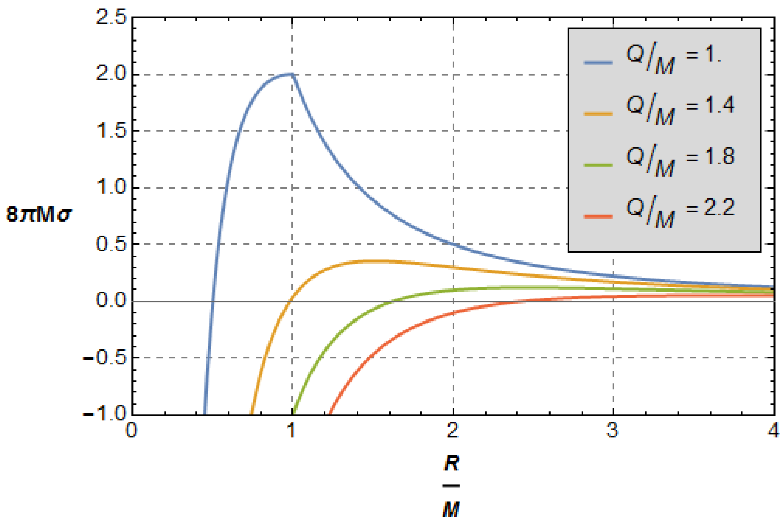
- Properties of a thin shell at region III of a non extremal Reissner-Nordström spacetime



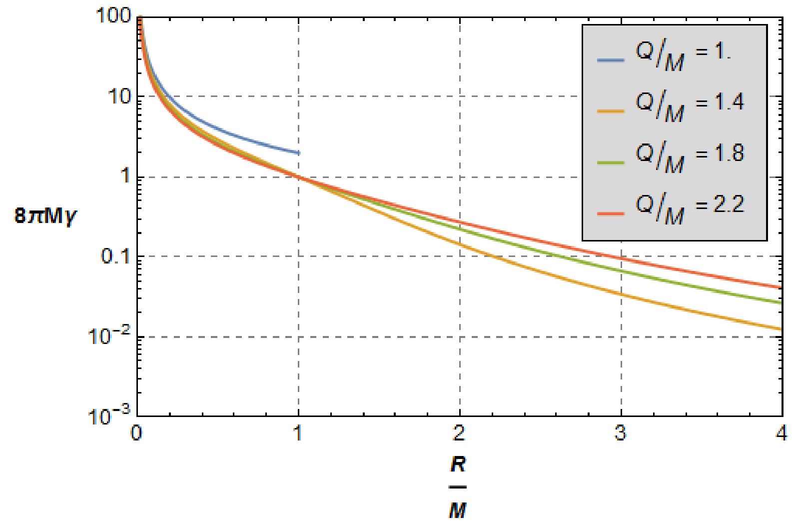




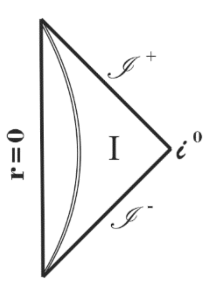
Extremal and Overcharged Reissner-Nordström Spacetime

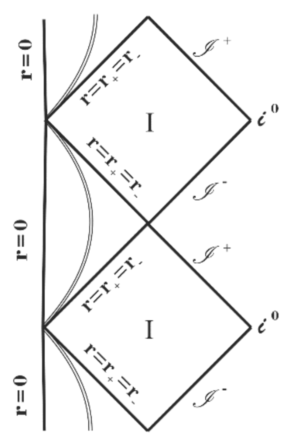


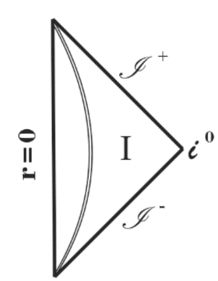
Minkowski – Extremal Reissner-Nordström



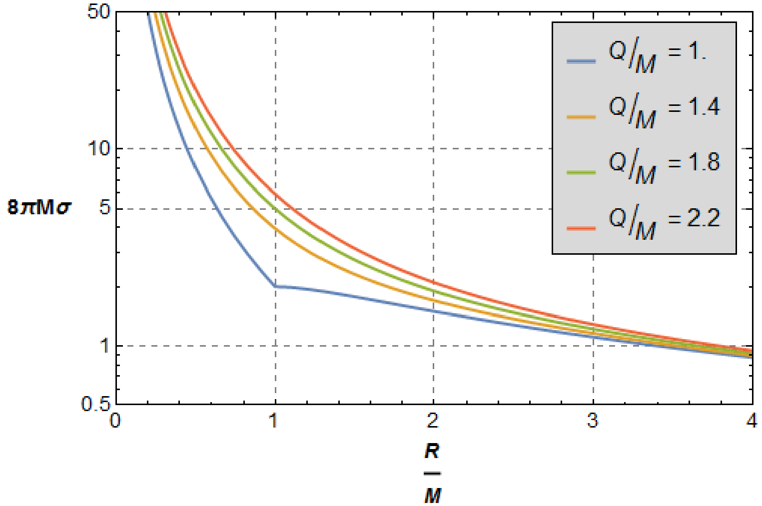
Minkowski – Overcharged Reissner-Nordström

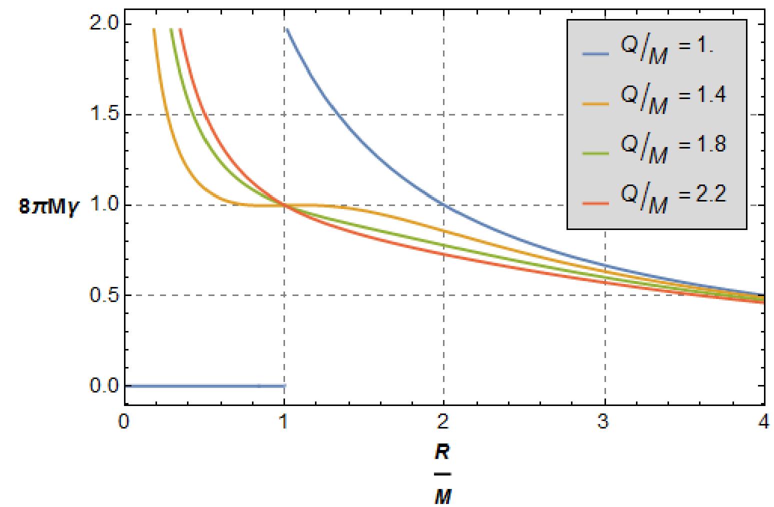




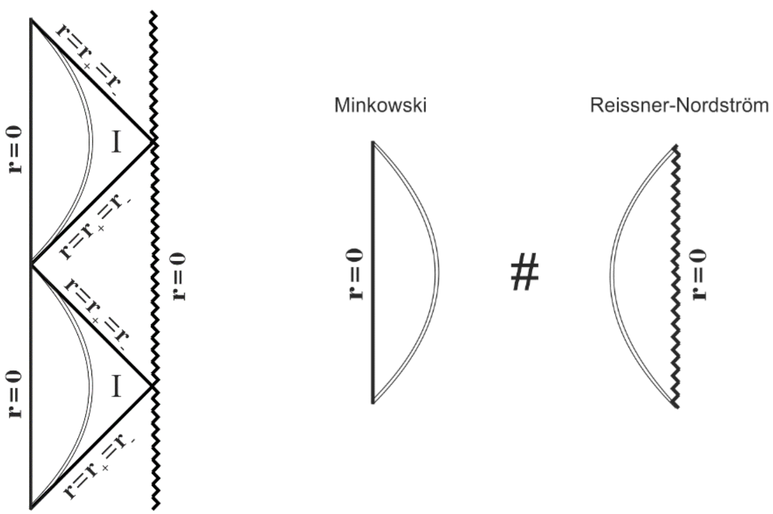


Extremal and Overcharged Reissner-Nordström Spacetime

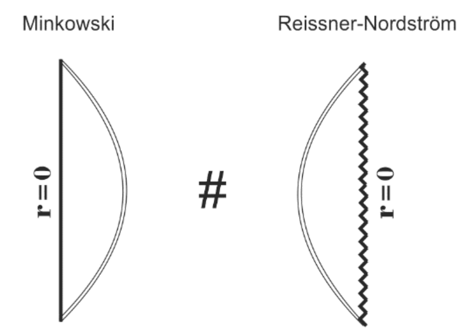




Minkowski – Extremal Reissner-Nordström

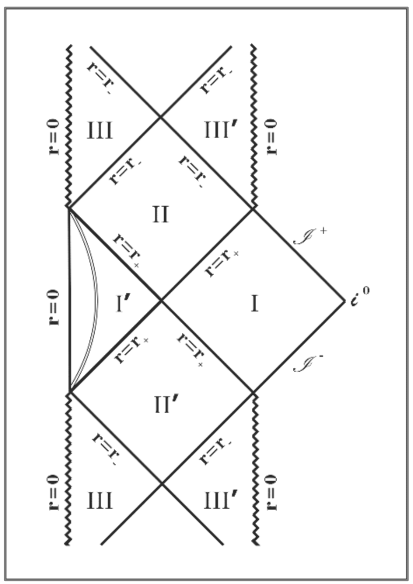


Minkowski – Overcharged Reissner-Nordström

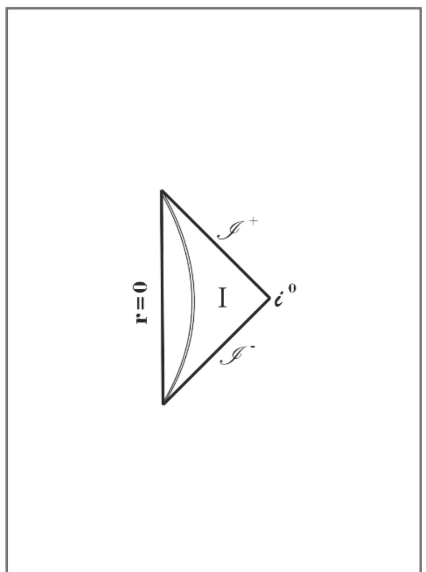


Energy Conditions

Type \ Energy Condition		Null energy condition	Weak energy condition	Dominant energy condition	Strong energy condition
Non-Extremal	a)	$R \geq R_{\mathrm{I}'}$	$R \geq R_{\mathrm{I}'}$	$R \geq R_{\mathrm{I}'}$	Never verified
	b)	$R > r_+$	$R > r_+$	$R \geq R_{\mathrm{I}}$	$R > r_+$
	c)	$0 < R < r_-$	$0 < R < r_-$	$0 < R \leq R_{\mathrm{III}}$	$0 < R < r_-$
	d)	Never verified	Never verified	Never verified	Never verified



a)



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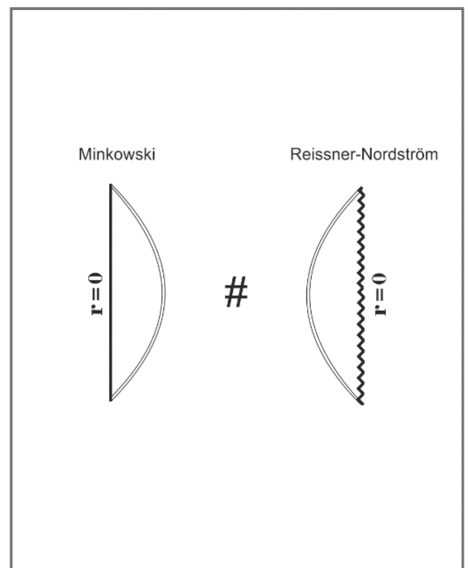

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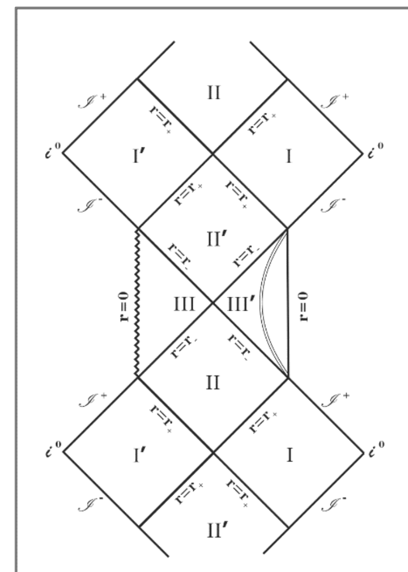
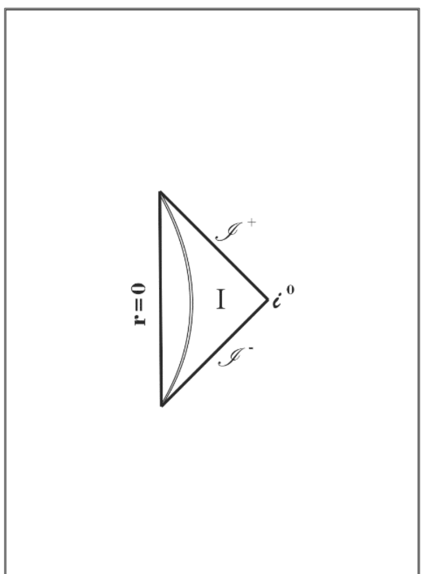


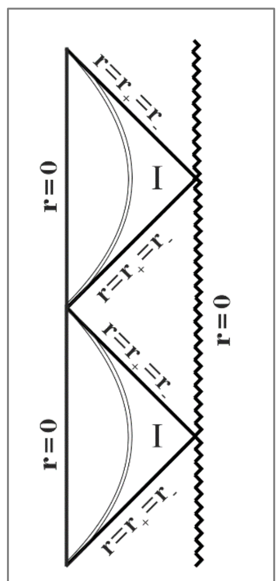
d)

Energy Conditions

Type \ Energy Condition		Null energy condition	Weak energy condition	Dominant energy condition	Strong energy condition
Extremal	a)	$R > r_+$	$R > r_+$	$R > r_+$	$R > r_+$
	b)	$R > r_+$	$R > r_+$	$R > r_+$	Never verified
	c)	Never verified	Never verified	Never verified	Never verified
	d)	$0 < R < r_+$	$0 < R < r_+$	$0 < R < r_+$	$0 < R < r_+$



a)



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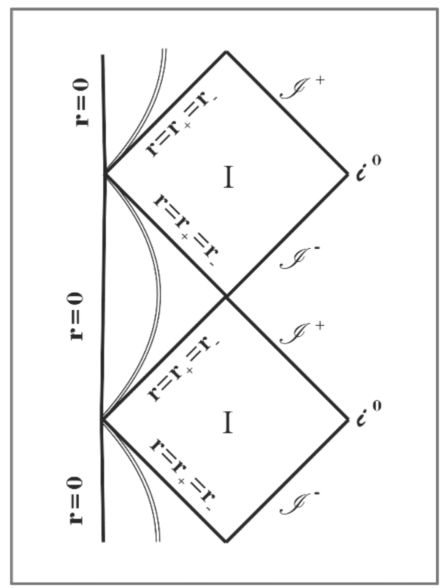

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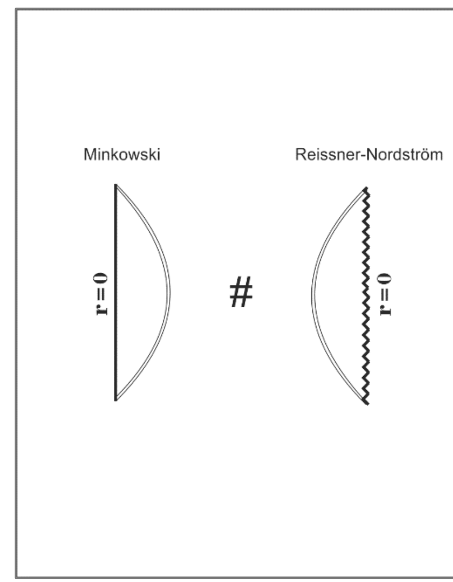
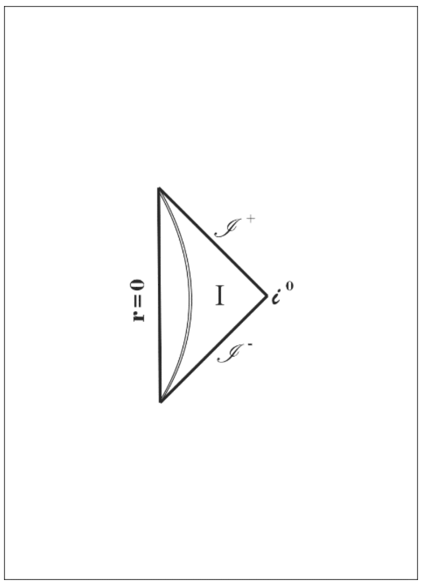


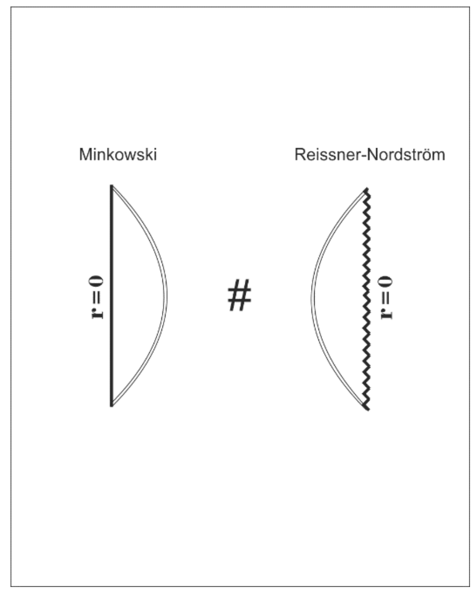
d)

Energy Conditions

Type \ Energy Condition		Null energy condition	Weak energy condition	Dominant energy condition	Strong energy condition
Overcharged	a)	$R \geq R_{\mathrm{I}'}$	$R \geq R_{\mathrm{I}'}$	$R \geq R_{\mathrm{I}'}$	$R \geq R_{\mathrm{OI}}$
	b)	$R > 0$	$R > 0$	$R > 0$	$R \leq R_{\mathrm{OI}'}$



a)



b)