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Dark energy fingerprints in the non-minimal Wu-Yang wormhole structure

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General formalism

The three-parameter non-minimal Einstein-Yang-Mills theory can be formulated in terms of the action functional

$$S_{\mathrm{NMEM}} = \int d^4x \sqrt{-g}\, \left[\frac{R}{16\pi} + \mathcal{L}_{(DE)} + \frac{1}{4} F^a_{ik} F^{ik\,a} + \frac{1}{4} \mathcal{R}^{ikmn} F^a_{ik} F^a_{mn} \right] , \qquad (1)$$

$$\mathcal{R}^{ikmn} \equiv \frac{q_1}{2} R \left(g^{im} g^{kn} - g^{in} g^{km} \right) + \frac{q_2}{2} \left(R^{im} g^{kn} - R^{in} g^{km} + R^{kn} g^{im} - R^{km} g^{in} \right) + q_3 R^{ikmn} , \qquad (2)$$

where q_1, q_2, q_3 are the phenomenological parameters describing the non-minimal coupling of electromagnetic and gravitational fields, the group indices a, b, c run from 1 to 3, the term $\mathcal{L}_{(DE)}$ is the Lagrangian describing the dark energy.

Equations

The variation of the action functional with respect to potential A_i^a yields

$$D_k \left(F^{ik\,a} + \mathcal{R}^{ikmn} F^a_{mn} \right) = 0, \qquad D_k F^a_{mn} \equiv \nabla_k F^a_{mn} + \epsilon_{abc} A^b_k F^c_{mn}. \tag{3}$$

The variation of the action with respect to the metric yields

$$R_{ik} - \frac{1}{2} R \, g_{ik} = 8\pi \, T^{(\text{eff})}_{ik} + 8\pi \, T^{(DE)}_{ik} \,. \tag{4}$$

The effective stress-energy tensor $T^{(\text{eff})}_{ik}$ can be divided into four parts:

$$T^{(\text{eff})}_{ik} = T^{(YM)}_{ik} + q_1 T^{(I)}_{ik} + q_2 T^{(II)}_{ik} + q_3 T^{(III)}_{ik} \,. \tag{5}$$

The first term $T^{(YM)}_{ik}$:

$$T^{(YM)}_{ik} \equiv \frac{1}{4} g_{ik} F^a_{mn} F^{mn\,a} - F^a_{in} F_k^{\ n\,a} \,, \tag{6}$$

is a stress-energy tensor of the pure Yang-Mills field.

The definitions of other three tensors are related to the corresponding coupling constants q_1, q_2, q_3:

$$T^{(I)}_{ik} = R\,T^{(YM)}_{ik} - \frac{1}{2}R_{ik}F^a_{mn}F^{mn\,a} + \frac{1}{2}\left[\nabla_i\nabla_k - g_{ik}\nabla^l\nabla_l\right]\left[F^a_{mn}F^{mn\,a}\right], \qquad (7)$$

$$T^{(II)}_{ik} = -\frac{1}{2}g_{ik}\left[\nabla_m\nabla_l\left(F^{mn\,a}F^{l\,a}_{\ \ n}\right) - R_{lm}F^{mn\,a}F^{l\,a}_{\ \ n}\right] - F^{ln\,a}\left(R_{il}F^a_{kn} + R_{kl}F^a_{in}\right) -$$

$$- R^{mn}F^a_{im}F^a_{kn} - \frac{1}{2}\nabla^m\nabla_m\left(F^a_{in}F_k^{\ n\,a}\right) + \frac{1}{2}\nabla_l\left[\nabla_i\left(F^a_{kn}F^{ln\,a}\right) + \nabla_k\left(F^a_{in}F^{ln\,a}\right)\right], \qquad (8)$$

$$T^{(III)}_{ik} = \frac{1}{4}g_{ik}R^{mnls}F^a_{mn}F^a_{ls} - \frac{3}{4}F^{ls\,a}\left(F_i^{\ n\,a}R_{knls} + F_k^{\ n\,a}R_{inls}\right) -$$

$$- \frac{1}{2}\nabla_m\nabla_n\left[F_i^{\ n\,a}F_k^{\ m\,a} + F_k^{\ n\,a}F_i^{\ m\,a}\right]. \qquad (9)$$

Non-minimal Dirac monopoles

We assume that electric charge is absent, $Q = 0$. The Yang-Mills equations are satisfied identically, when the Yang-Mills field strength tensor F^a_{ik} outside a point-like magnetic charge ν has the form

$$F^a_{ik} = \frac{\nu\, x^a}{r} \cdot \sin\theta \left(\delta^\theta_i \delta^\varphi_k - \delta^\theta_k \delta^\varphi_i\right) , \tag{10}$$

$$x^1 = r\cos\varphi \sin\theta, \quad x^2 = r\sin\varphi\sin\theta, \quad x^3 = r\cos\theta.$$

These quantities depend neither on the radial variable r, nor on the coupling parameters q_1, q_2, q_3. Thus, the well-known solution with a monopole-type magnetic field satisfies the non-minimally extended Yang-Mills equations.

Let us consider a static spherically symmetric spacetime with the metric

$$ds^2 = \sigma^2 N dt^2 - \frac{dr^2}{N} - R^2(r)\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad r \in (-\infty; +\infty) \tag{11}$$

Here σ, N, and $R(r)$ are functions depending on the radial coordinate r only.

We will consider the simplest variant to introduce the dark energy, namely, as a cosmological Λ-term:

$$T_{ik}^{(DE)} = \frac{\Lambda}{8\pi} g_{ik}. \tag{12}$$

Key gravitational field equations

For this metric, there exist only two independent equations:

$$\left(1-\frac{\kappa q_1}{R^4}\right)\left[\frac{\sigma' R'}{\sigma R}-\frac{R''}{R}\right]=\frac{\kappa(10q_1+4q_2+q_3)R'^2}{R^6}, \tag{13}$$

$$\frac{1-NR'^2}{R^2}-\left(1-\frac{\kappa q_1}{R^4}\right)\left(\frac{N'R'}{R}+\frac{2NR''}{R}\right)=$$
$$=\frac{\kappa}{R^4}\left\{\frac{1}{2}-\frac{q_1+q_2+q_3}{R^2}+\frac{(13q_1+4q_2+q_3)NR'^2}{R^2}\right\}+\Lambda\,. \tag{14}$$

The parameter κ is defined as $\kappa = 8\pi\nu^2$. The prime denotes the derivative with respect to the variable r.

Direct integration of (13) gives us σ as a function of $R(r)$ and its derivative as follows:

$$\sigma=\sigma_0 R'\left(\frac{R^4-\kappa q_1}{R^4}\right)^{\beta}, \quad \beta\equiv\frac{10q_1+4q_2+q_3}{4q_1}\,. \tag{15}$$

Wormhole-type solution

We are interested in the analysis of wormhole-type solutions.

Requirements:
1) $R^{'}(0) = 0$, $R''(0) \neq 0$ (throat conditions),
2) $\sigma(r)$ and $N(r)$ are regular functions.

It is possible, when three non-minimal coupling parameters q_1, q_2, and q_3 are connected with the wormhole throat radius $R(0) = a$ by the following relationships:

$$q_1 = \frac{a^4}{\kappa}\,, \quad q_2 = -\frac{10a^4}{3\kappa} - \frac{a^2}{6} - \frac{\Lambda}{3\kappa}a^6\,,$$
$$q_3 = \frac{4a^4}{3\kappa} + \frac{2a^2}{3} + \frac{4\Lambda}{3\kappa}a^6\,. \qquad (16)$$

Wormhole-type solution

As a result, we obtain the regular solution for the metric functions:

$$\sigma(r) = \sigma_0 \frac{R' R^2}{\sqrt{R^4 - a^4}}\,, \tag{17}$$

$$N(r) = \frac{R(r)}{R'^2(r)\sqrt{R^4 - a^4}} \int\limits_0^r \frac{R'(x)dx}{R^2(x)} \sqrt{\frac{R^2 - a^2}{R^2 + a^2}} \left[(R^2 + a^2)(1 - \Lambda R^2) - \left(\frac{\kappa}{2} + \Lambda a^4 \right) \right]. \tag{18}$$

As for the integration constant σ_0, if we assume that $R^{'}(r \to \infty) \to 1$, we can put σ_0=1, providing $\sigma(\infty)$=1.

Since now

$$a R''(0) N(0) = \frac{1}{3} - \frac{\kappa}{12a^2} - \frac{\Lambda a^2}{2}\,, \tag{19}$$

the wormhole is traversable, when

$$N(0) > 0 \quad \Rightarrow \quad \Lambda < \frac{2}{3a^2}\left(1 - \frac{\kappa}{4a^2} \right). \tag{20}$$

The function $R(r)$ can be chosen according to physical requirements; when we consider the wormhole-type solutions, we assume that

$$R(0) = a > 0\,, \quad R'(0) = 0\,, \quad R''(0) > 0\,, \tag{21}$$

where a is the throat radius.

The most known function satisfying these conditions is

$$R(r) = \sqrt{r^2 + a^2}.$$

We will use it below for the reconstruction of exact solutions to the master equations with constant dark energy pressure; this radial function displays the asymptotic behavior $R(r \to \infty) \to r$.

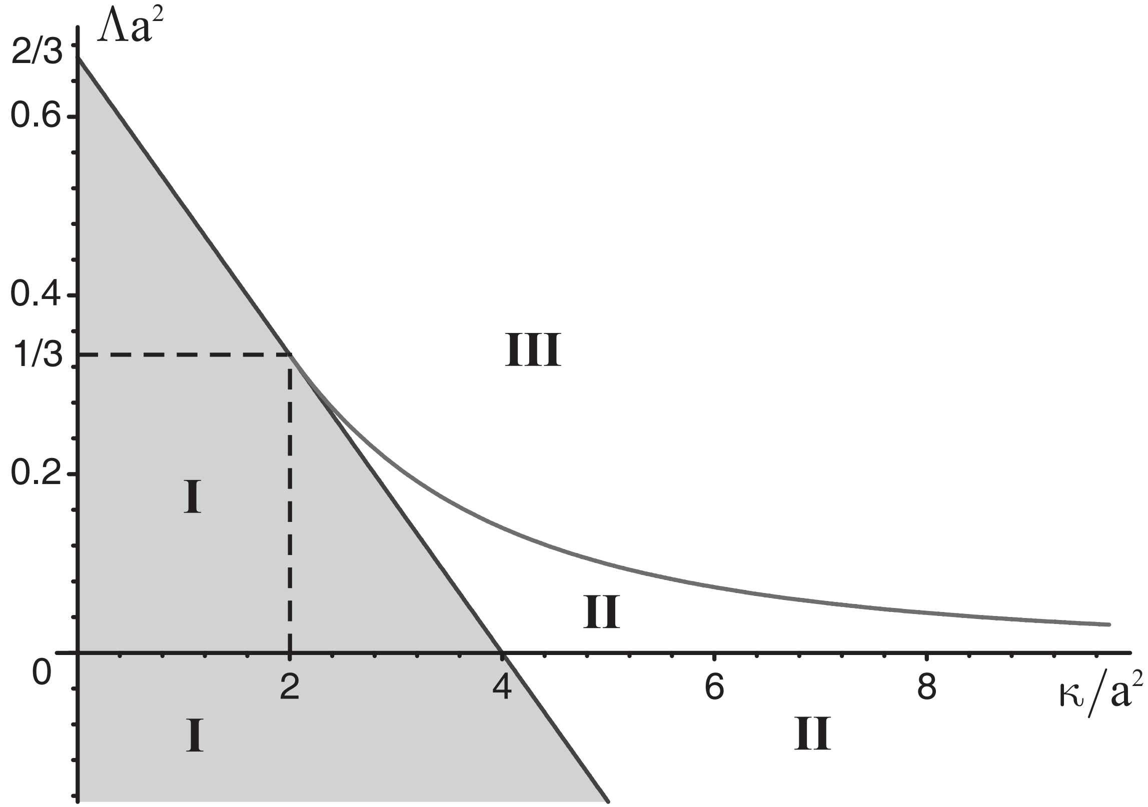
The metric function $\sigma(r)$ takes the form

$$\sigma(r) = \sqrt{\frac{r^2+a^2}{r^2+2a^2}}\,. \tag{22}$$

The function $\sigma(r)$ reaches neither zero nor infinite values, thus, the causal structure of the wormhole is predetermined only by the properties of the function $N(r)$, which can be written as follows:

$$N(r) = \frac{(r^2+a^2)^{3/2}}{r^3\sqrt{r^2+2a^2}} \int\limits_0^r \frac{x^2dx}{(x^2+a^2)^{3/2}\sqrt{x^2+2a^2}} \times \\ \times \left[-\Lambda x^4 + x^2(1-3\Lambda a^2) + (2a^2 - \frac{\kappa}{2} - 3\Lambda a^4)\right]. \tag{23}$$

Clearly, we deal with three-parameter family of regular solutions: the metric function N depends on the cosmological constant Λ, on the gauge charge $\kappa \equiv 8\pi\nu^2$, and on the parameter of the non-minimal coupling q_1 through the throat radius $a \equiv (\kappa q_1)^{\frac{1}{4}}$. Horizons are known to appear at $r = r_{(\mathrm{s})}$, where $r_{(\mathrm{s})}$ are the zeroes of the metric function, i.e., $N(r_{(\mathrm{s})})$=0.



Domains on the semiplane of the parameters $\frac{\kappa}{a^2} > 0$ and Λa^2, in which the Λ-influenced non-minimal Wu-Yang wormhole has a traversable or non-traversable throats. Domain I (shaded) indicates wormholes with a traversable throat, i.e., when $N(0) > 0$. Domain II relates to the spacetimes with two R-regions and a T-region between them; the throat is non-traversable in this case. Domain III corresponds to the spacetimes without R-regions.

Thank you for your attention!