Local non-negative initial data scalar characterisation of the Kerr solution

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Bibliography:

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Outline

The role of the Kerr solution

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- The non-linear stability of the domain of outer communication of the Kerr solution. Requires a definition of vacuum initial data *close* to *Kerr initial data*.

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Both definitions can be encompassed within the notion of *non-Kerrness*.

non-Kerrness $\equiv$ non-negative scalar characterising vacuum initial data *close* to Kerr initial data.

Kerr initial data

Definition

A vacuum initial data set (Σ, h_{ij}, K_{ij}) is called Kerr initial data or a Kerr initial data set if there exists an isometric embedding $\phi : \Sigma \to \mathcal{M}$ where $\mathcal{M}$ is an open subset of the Kerr spacetime.

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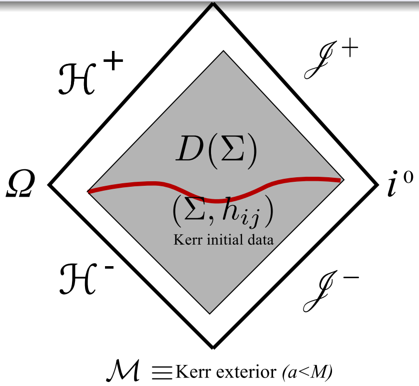

Non-Kerrness

Definition

Let $\mathcal{M}$ be a vacuum space-time and Σ a co-dimension 1 Riemannian manifold and $\phi : \Sigma \to \mathcal{M}$ an isometric embedding. A *non-Kerrness* is any non-negative quantity $\rho(\Sigma)$ which vanishes if and only if $\mathcal{M}$ is an open subset of the Kerr spacetime.

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- If (Σ, h_{ij}, K_{ij}) is a vacuum initial data set, use $\rho(\Sigma)$ to define when the data are *close* to Kerr initial data.

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- If (Σ, h_{ij}, K_{ij}) is a vacuum initial data set, use $\rho(\Sigma)$ to define when the data are *close* to Kerr initial data.
- To test the asymptotic evolution towards the Kerr solution (for example in a numerical simulation) one can divide up the space-time into space-like slices $\{\Sigma_t\}$ and check whether these approach a slice in the Kerr space-time. Use the function $f(t) \equiv \rho(\Sigma_t)$ for that.

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Previous work

Explicit construction of Kerr initial data:

- A. García-Parrado and J. A. Valiente Kroon, *Kerr initial data*, Class. Quantum Grav. **25**, (2008), 205018.

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Our result

Theorem

Let (Σ, h_{ij}, K_{ij}) *be a vacuum initial data set fulfilling certain regularity conditions to be detailed later. Then we can construct a non-negative scalar* $\mathcal{L}(h_{ij}, K_{ij}) \geq 0$ *at each point of* Σ *defined exclusively in terms of* h_{ij}*,* K_{ij} *and their covariant derivatives (with respect to the connection compatible with* h_{ij}*) such that*

$$\mathcal{L}(h_{ij}, K_{ij}) = 0 \Longleftrightarrow (\Sigma, h_{ij}, K_{ij}) \text{ are Kerr initial data.}$$

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We follow the terminology laid by previous authors and call the scalar $\mathcal{L}(h_{ij}, K_{ij})$ the *non-Kerrness*.

A local invariant characterisation of the Kerr solution

Theorem (Ferrando & Sáez (2009))

A solution $(\mathcal{M}, g_{\mu\nu})$ of the vacuum Einstein field equations is locally isometric to the Kerr spacetime if and only if the following conditions hold in an open set of $\mathcal{M}$

$$A^2 + B^2 \neq 0 \,, \quad Q_{\mu\nu\lambda\rho}\nabla^\mu B \nabla^\lambda B \neq 0 \,, \quad \sigma > 0 \,,$$
$$\frac{1}{2} C^{\sigma\tau}{}_{\mu\nu} C_{\sigma\tau\lambda\rho} + \alpha C_{\mu\nu\lambda\rho} + \beta C^*_{\mu\nu\lambda\rho} - \frac{1}{3}\left(A G_{\mu\nu\lambda\rho} - B\eta_{\mu\nu\lambda\rho}\right) = 0,$$
$$Q_{\mu\nu\lambda\rho}\nabla^\mu A \nabla^\lambda A + Q_{\mu\nu\lambda\rho}\nabla^\mu B \nabla^\lambda B = 0,$$
$$(1 - 3\lambda^2)\beta + \lambda(3 - \lambda^2)\alpha = 0 \,,$$

and there exists a vector field ξ^μ fulfilling the properties

$$\Xi_{\mu\nu} = \left(\frac{\alpha}{1 - 3\lambda^2}\right)^{\frac{2}{3}} \xi_\mu \xi_\nu \,, \quad \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0.$$

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All the conditions in the theorem are written in terms of *concomitants* of the Weyl tensor.

Orthogonal splitting of Ferrando & Sáez (FS) result

- **Idea:** FS conditions are made of *concomitans* of the Weyl tensor. Compute their orthogonal splitting and find their projection to the initial data hypersurface Σ. This yields *necessary conditions* which a Kerr initial data set must satisfy.

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- Take for instance the Weyl tensor $C_{\mu\nu\lambda\sigma}$

$$C_{\mu\nu\lambda\sigma} = 2\left(l_{\mu[\lambda}E_{\sigma]\nu} - l_{\nu[\lambda}E_{\sigma]\mu} - n_{[\lambda}B_{\sigma]\tau}\varepsilon^{\tau}{}_{\mu\nu} - n_{[\mu}B_{\nu]\tau}\varepsilon^{\tau}{}_{\lambda\sigma}\right)$$

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- Weyl tensor electric & magnetic parts

$$E_{\tau\sigma} \equiv C_{\tau\nu\sigma\lambda}n^{\nu}n^{\lambda}, \quad B_{\tau\sigma} \equiv C^{*}_{\tau\nu\sigma\lambda}n^{\nu}n^{\lambda},$$

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$$E_{\tau\sigma} \equiv C_{\tau\nu\sigma\lambda}n^{\nu}n^{\lambda}, \quad B_{\tau\sigma} \equiv C^{*}_{\tau\nu\sigma\lambda}n^{\nu}n^{\lambda},$$

- These can be related to the vacuum initial data h_{ij}, K_{ij}

$$E_{ij} = r_{ij} + KK_{ij} - K_{ik}K^{k}{}_{j} \; ,$$
$$B_{ij} = \epsilon^{kl}{}_{(i}D_{|k}K_{l|j)}.$$

Killing initial data

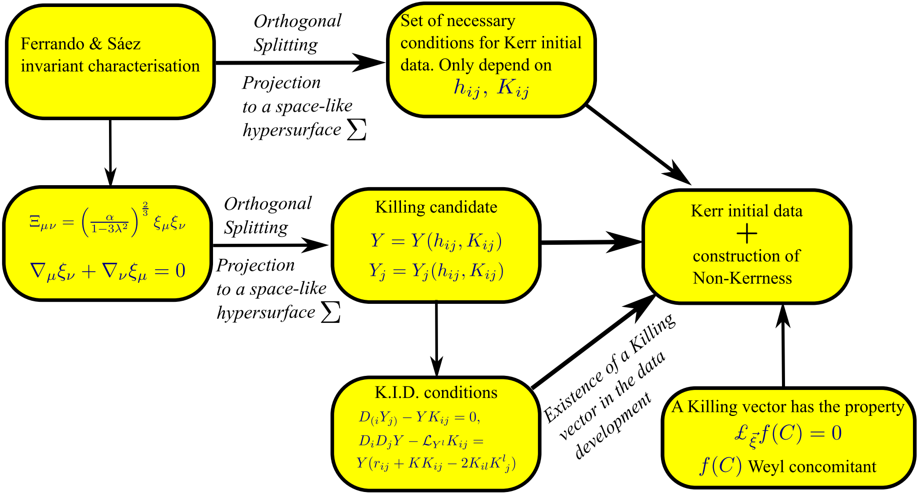
To find sufficient conditions fulfilled by a Kerr initial data set we use the Killing vector ξ^μ to propagate the necessary conditions found in the previous step. To achieve this we need to find conditions for the existence of the Killing vector ξ^μ.

Theorem (Killing initial data (KID))

The necessary and sufficient condition for there to exist a Killing vector field ξ^μ *in the data development of a vacuum initial data set* (Σ, h_{ij}, K_{ij}) *is that a pair* (Y, Y_j) *defined on* Σ *fulfills*

$$D_{(i}Y_{j)} - YK_{ij} = 0,$$
$$D_iD_jY - \mathcal{L}_{Y^l}K_{ij} = Y(r_{ij} + KK_{ij} - 2K_{il}K^l{}_j).$$

Construction of a local *non-Kerrness*



Main Theorem

Theorem

Let (Σ, h_{ij}, K_{ij}) *be a vacuum initial data set and assume that on* Σ *the data fulfills the properties*

$$\sigma > 0\ ,$$
$$\mathcal{K} \equiv \big(\mathfrak{r}(B)^2 + \mathfrak{j}(B)_i\mathfrak{j}(B)^i + \mathfrak{t}(B)_{ij}\mathfrak{t}(B)^{ij}\big)(A^2 + B^2) > 0\ ,$$

Under these conditions define the following non-negative scalar (all the variables are defined in terms of h_{ij}, K_{ij}*)*

$$\mathcal{L}(h_{ij}, K_{ij}) \equiv \frac{(\mathfrak{r}(A) + \mathfrak{r}(B))^2 + (\mathfrak{j}(A)_i + \mathfrak{j}(B)_i)(\mathfrak{j}(A)^i + \mathfrak{j}(B)^i)}{\sigma^{14}} + \frac{(\mathfrak{t}(A)_{ij} + \mathfrak{t}(B)_{ij})(\mathfrak{t}(A)^{ij} + \mathfrak{t}(B)^{ij})}{\sigma^{14}} + \frac{\mathfrak{a}_{ij}\mathfrak{a}^{ij} + \mathfrak{b}_{ij}\mathfrak{b}^{ij}}{\sigma^4} + \frac{\big((1-3\lambda^2)\beta + \lambda(3-\lambda^2)\alpha\big)^2}{\sigma^2} + \frac{(\mathfrak{B}_{ij}\mathfrak{B}^{ij})^3}{\sigma^4} + \frac{(\mathfrak{C}_{ij}\mathfrak{C}^{ij})^3}{\sigma^7} + \frac{\Omega}{\sigma^2}.$$

The scalar $\mathcal{L}(h_{ij}, K_{ij})$ *vanish if and only if* (Σ, h_{ij}, K_{ij}) *are Kerr initial data.*

Open issues

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- The propagation problem: since $\mathcal{L}(h_{ij}, K_{ij})$ only depend on the initial data h_{ij}, K_{ij} one can obtain the evolution of $\mathcal{L}(h_{ij}, K_{ij})$ under the Einstein's vacuum equations.
- It has been always assumed that Σ is space-like. Generalise the analysis for Σ of arbitrary causal character.