Stability and the effective metric

Santiago Esteban Perez Bergliaffa
Department of Theoretical Physics
Institute of Physics
University of the State of Rio de Janeiro

IST 2015

Effective metric for a noninear scalar theory

$$S[\phi] = \int \sqrt{-g}\mathcal{L}(W)d^4x,$$

$$W = g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$$

Background metric

EOM:

$$\mathcal{L}_W \,\Box\phi + \partial^\mu\phi(\partial_\mu W)\mathcal{L}_{WW} = 0$$

$$\phi = \phi_0 + \varepsilon\phi_1$$

$$\rightarrow$$

$$(\sqrt{-g}(\mathcal{L}_W g^{\mu\upsilon} + 2\mathcal{L}_{WW} g^{\mu\alpha}\phi_{0,\alpha} g^{\nu\beta}\phi_{0,\beta})\phi_{1,\mu})_{,\nu} = 0.$$

+ eikonal approx.

$$\sqrt{-g}\left(\mathcal{L}_W g^{\mu\nu} + 2\mathcal{L}_{WW} g^{\mu\alpha}\phi_{0,\alpha} g^{\nu\beta}\phi_{0,\beta}\right) = \sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu},$$

"Effective metric"

(all quantities evaluated at the background solution)

Goulart and SEPB, 2011

$$(\sqrt{-g}(\mathcal{L}_W g^{\mu\nu} + 2\mathcal{L}_{WW} g^{\mu\alpha}\phi_{0,\alpha} g^{\nu\beta}\phi_{0,\beta})\phi_{1,\mu})_{,\nu} = 0.$$

$$\sqrt{-g}\left(\mathcal{L}_W g^{\mu\nu} + 2\mathcal{L}_{WW} g^{\mu\alpha}\phi_{0,\alpha} g^{\nu\beta}\phi_{0,\beta}\right) = \sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu},$$

$$(\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\phi_{1,\mu})_{,\nu} = 0.$$

In the linear case, $\mathcal{L}(W,\phi) = W$ the effective metric reduces to the background metric.

In the case of theories with more degrees of freedom there can be birefringence and/or bimetricity.

Goulart and SEPB, 2009

From

$$(\sqrt{-\widetilde{g}}\,\widetilde{g}^{\mu\nu}\phi_{1,\mu})_{,\nu} = 0.$$

the action for the (high-energy) perturbations is

$$S_2 = \int \sqrt{-\widetilde{g}}\,\widetilde{g}^{\mu\nu}\phi^{4}_{1,\mu}\phi_{1,\nu}d^3x.$$

$$\widetilde{T}_{\mu\nu} = \frac{\delta S_2}{\delta\widetilde{g}^{\mu\nu}}.$$

→

$$\widetilde{T}^{\mu}_{\nu} = \widetilde{g}^{\mu\lambda}\phi_{1,\lambda}\phi_{1,\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\widetilde{g}^{\alpha\beta}\phi_{1,\alpha}\phi_{1,\beta}$$

$$\widetilde{\nabla}_{\mu}\widetilde{T}^{\mu\nu} = 0.$$

Linear stability using the effective metric

(Moncrief, 1980, for a test perfect fluid in potential flux accreting onto a Schwarzschild black hole)

X^μ is a Killing vector of the background metric (hence of the eff. metric) →

$$\widetilde{\nabla}_\mu \left(X^\nu \widetilde{T}^\mu_\nu \right) = 0,$$

$$\partial_\nu \left(\sqrt{-\tilde{g}} \widetilde{X}^\mu \widetilde{T}^\nu_\mu \right) = 0$$

Integrating in a 3-volume V

$$\widetilde{X}^\nu = \delta^\nu_t$$

$$\int_V \partial_\nu (\sqrt{-\tilde{g}} \widetilde{T}^\nu_t) d^3x = 0.$$

$$\int_V \partial_t (\sqrt{-\tilde{g}} \widetilde{T}^t_t) d^3x + \int_V \partial_i (\sqrt{-\tilde{g}} \widetilde{T}^i_t) d^3x = 0.$$

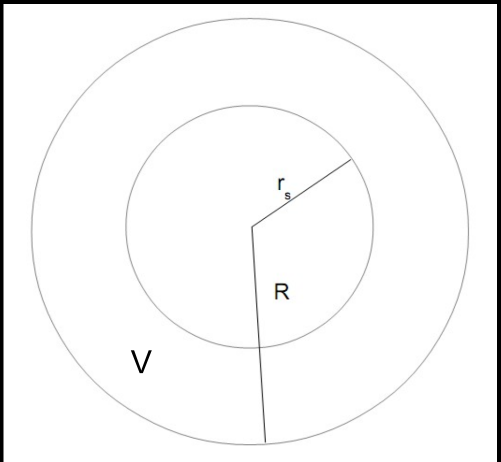
$$\int_V \partial_t(\sqrt{-\tilde{g}}\widetilde{T}_t^t)d^3x + \int_V \partial_i(\sqrt{-\tilde{g}}\widetilde{T}_t^i)d^3x = 0.$$

$$\widetilde{E} = \int_V \sqrt{-\widetilde{g}}\ \widetilde{T}_t^t d^3x.$$

$$\frac{d\widetilde{E}}{dt} = -\int_S dS_i \sqrt{-\widetilde{g}}\ \widetilde{T}_t^i.$$

Linear stability ⟷ $\frac{d\widetilde{E}}{dt} \leq 0$ **and** $\widetilde{E} > 0$

$$\widetilde{T}^\mu_\nu = \widetilde{g}^{\mu\lambda}\phi_{1,\lambda}\phi_{1,\nu} - \frac{1}{2}\delta^\mu_\nu\widetilde{g}^{\alpha\beta}\phi_{1,\alpha}\phi_{1,\beta}$$



$$\widetilde{T}^{\mu}_{\nu} = \widetilde{g}^{\mu\lambda}\phi_{1,\lambda}\phi_{1,\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\widetilde{g}^{\alpha\beta}\phi_{1,\alpha}\phi_{1,\beta}$$

$$\frac{d\widetilde{E}}{dt} = I_1 + I_2,$$

$$I_1 = -\oint_{S_R} \sqrt{-\tilde{g}}\left(\phi_{1,r}\phi_{1,t}\widetilde{g}^{rr} + (\phi_{1,t})^2\widetilde{g}^{rt}\right)\Big|_R \, dS_r,$$

$$I_2 = \oint_{r_s} \sqrt{-\tilde{g}}\left(\phi_{1,r}\phi_{1,t}\widetilde{g}^{rr} + (\phi_{1,t})^2\widetilde{g}^{rt}\right)\Big|_{r_s} \, dS_r,$$

r_s : **“sonic horizon”**

$$\widetilde{E} = \int_V \sqrt{-\widetilde{g}}\ \widetilde{T}^t_t d^3x.$$

$$\widetilde{T}^\mu_\nu = \widetilde{g}^{\mu\lambda}\phi_{1,\lambda}\phi_{1,\nu} - \frac{1}{2}\delta^\mu_\nu\widetilde{g}^{\alpha\beta}\phi_{1,\alpha}\phi_{1,\beta}$$

H1)

$$\sqrt{-\widetilde{g}}\widetilde{g}^{rr} \underset{r\to\infty}{\to} \sqrt{-g}g^{rr} = r^2\sin\theta,$$

$$\tilde{g}^{rt} \underset{r\to\infty}{\longrightarrow} 0$$

H2) The perturbations have finite energy:

$$\phi_{1,t} \sim \frac{a}{r^{\frac{3}{2}+\epsilon}}; \quad \phi_{1,r} \sim \frac{a'}{r^{\frac{5}{2}+\epsilon}}, \quad \epsilon>0, \quad r\to\infty$$

$$I_1 = -\oint_{S_R} \sqrt{-\tilde{g}}\left(\phi_{1,r}\phi_{1,t}\widetilde{g}^{rr} + (\phi_{1,t})^2\widetilde{g}^{rt}\right)\Big|_{R\to\infty} dS_r, \Longrightarrow I_1 = 0$$

$$\frac{d\widetilde{E}}{dt} = I_2 = \oint\limits_{S_{r_s}} \sqrt{-\tilde{g}}\left(\phi_{1,r}\phi_{1,t}\widetilde{g}^{rr} + (\phi_{1,t})^2\widetilde{g}^{rt}\right)\Big|_{r_s} dS_r,$$

$$\widetilde{g}^{rr}(r_s) = 0 \quad \Longrightarrow \quad \frac{d\widetilde{E}}{dt} = \int \sqrt{-\tilde{g}}(\phi_{1,t})^2\widetilde{g}^{rt}\Big|_{r_s} dS_r.$$

Linear stability $\Longleftrightarrow$ $\dfrac{d\widetilde{E}}{dt} \leq 0$ $\qquad \widetilde{E} > 0$

Example: Frolov (2004)

$$\mathcal{L}(W) = \frac{1}{2}(W - A)^2.$$

"Effective cosmological constant"
(Arkani-Hamed et al, 2003)

$$ds^2 = f\,dt^2 - f^{-1}dr^2 - r^2 d\Omega^2,$$

$$f(r) = 1 - r_g/r$$

Stationary solution + spherical symmetry → $\phi = t + \psi(r),$

EOM $$\mathcal{L}_W \, \partial_r^* \psi = \alpha \frac{r_g^2}{r^2},$$

$$\partial_r^* \equiv f(r)\partial_r.$$

$$W = \frac{1 - (\partial_r^* \psi)^2}{f(r)},$$

$$\sqrt{-g}\left(\mathcal{L}_W g^{\mu\nu} + 2\mathcal{L}_{WW} g^{\mu\alpha}\phi_{0,\alpha} g^{\nu\beta}\phi_{0,\beta}\right) = \sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu},$$

$$M^{\mu\nu} = \mathcal{L}_W g^{\mu\nu} + 2\mathcal{L}_{WW} g^{\mu\alpha}\phi_{0,\alpha} g^{\nu\beta}\phi_{0,\beta}$$

$$\sqrt{-\tilde{g}} = (-g)\sqrt{M}$$

$$\tilde{g}^{\mu\nu} = \frac{M^{\mu\nu}}{\sqrt{-g}\sqrt{M}}$$

$$\mathcal{L}(W) = \frac{1}{2}(W - A)^2.$$

$$W = \frac{1 - (\partial_r^* \psi)^2}{f}$$

→ **_1 for r → ∞_**

$$\sqrt{-\tilde{g}}\tilde{g}^{rr} \underset{r\to\infty}{\rightarrow} \sqrt{-g}g^{rr} = r^2 \sin\theta,$$

$$\tilde{g}^{rt} \xrightarrow[r\to\infty]{} 0$$

Back to stability:

$$\frac{d\widetilde{E}}{dt} = \int \sqrt{-\tilde{g}}(\phi_{1,t})^2\tilde{g}^{rt}\Big|_{r_s} dS_r.$$

$$\mathcal{L}(W) = \frac{1}{2}(W - A)^2.$$

$$\phi = t + \psi(r),$$

$$\sqrt{-g}\,(\mathcal{L}_W g^{\mu\nu} + 2\mathcal{L}_{WW}\phi_0^{;\mu}\phi_0^{;\nu}) = \sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu},$$

↓

$$\sqrt{-\tilde{g}}\tilde{g}^{tr} = \sqrt{-g}2(-1)\psi_{,r}$$

$$\frac{d\widetilde{E}}{dt} = -2\int r^2 sen\theta\psi_{,r}\phi_{1,t}^2 d\theta d\varphi\Big|_{r_s}$$

(only the sign of $\psi_{,r}$ Is needed)

There is only one solution that goes from infinity (with null radial velocity) to r_g, and satisfies the condition $\psi_{,r} > 0$ (Frolov 2004)

→ The rhs is negative

We still need to prove that $\widetilde{E} > 0$

$$\widetilde{T}^t_t = \frac{1}{2}\widetilde{g}^{\,tt}(\partial_t\phi_1)^2 - \frac{1}{2}\widetilde{g}^{\,ij}(\partial_i\phi_1)(\partial_j\phi_1),$$

$$\widetilde{g}^{\,ij} = \mathrm{diag}(\widetilde{g}^{\,rr}, \widetilde{g}^{\,\theta\theta}, \widetilde{g}^{\,\varphi\varphi}),$$

$$\widetilde{g}^{\mu\nu} = \frac{M^{\mu\nu}}{\sqrt{-g}\sqrt{M}},$$

$$M^{\mu\nu} \equiv \mathcal{L}_W g^{\mu\nu} + 2\mathcal{L}_{WW}\Phi^{\mu\nu}\big|_0.$$

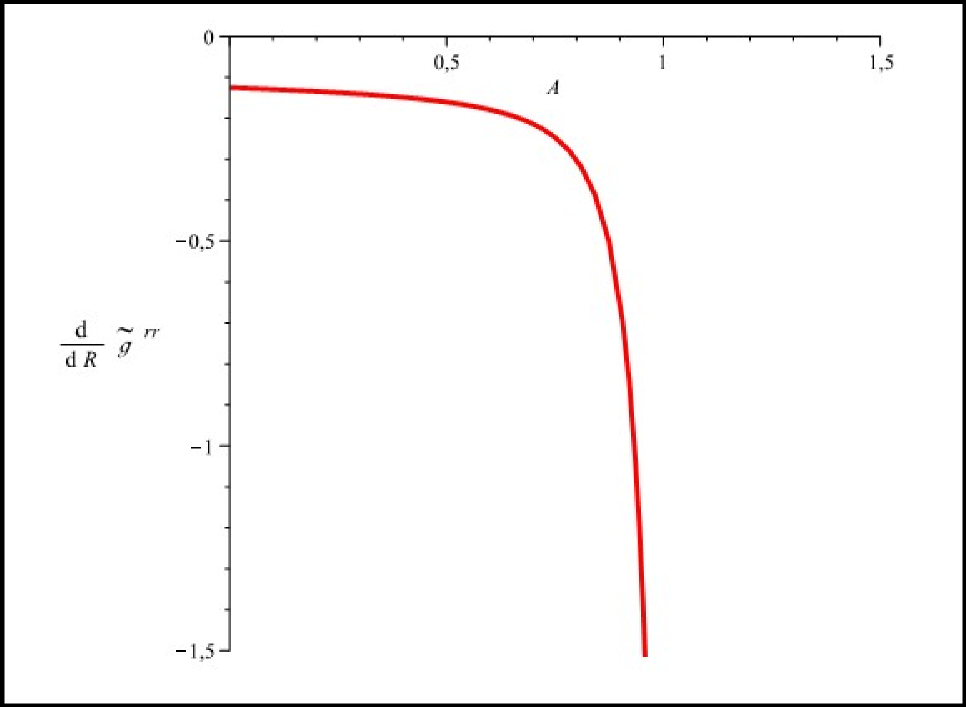
$$\Phi^{\mu\nu} \equiv g^{\mu\alpha}\phi_{,\alpha}g^{\nu\beta}\phi_{,\beta}.$$

$\mathcal{L}_W > 0$ $\quad$ $\mathcal{L}_{WW} = 1$ $\quad$ → **The *tt*, *θθ*, and *ϕϕ* components have the right sign.**

To prove that the *rr* component is negative outside the horizon:

1) It only has one zero (at the sonic horizon) $M^{rr}=3v^2+Af-1$ ***OK!***

2) Its derivative is negative at the sonic horizon:



→ The system is linearly stable under high-energy perturbations.

(C. A. Paz Rivasplata, J. M. Salim, SEPB, PRD 2014)

Stability using the effective potential

$$(\sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu}\phi_{1,\mu})_{,\nu} = 0.$$

$$\begin{aligned} dt &= dT - \frac{\tilde{g}_{rt}}{\tilde{g}_{tt}} dR, \\ dr &= dR. \end{aligned}$$

$$\sqrt{-g}\,(\mathcal{L}_W g^{\mu\nu} + 2\mathcal{L}_{WW}\phi_0^{;\mu}\phi_0^{;\nu}) = \sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu},$$

$$\tilde{G}^{tt} = \frac{\tilde{g}^{tt}\tilde{g}^{rr} - \tilde{g}^{rt}}{\tilde{g}^{rr}},$$

$$\tilde{G}^{rr} = \tilde{g}^{rr},$$

$$\tilde{G}^{\theta\theta} = \tilde{g}^{\theta\theta},$$

$$\tilde{G}^{\varphi\varphi} = \tilde{g}^{\varphi\varphi}.$$

$$\tilde{G}^{rr}(r_s) = 0,$$

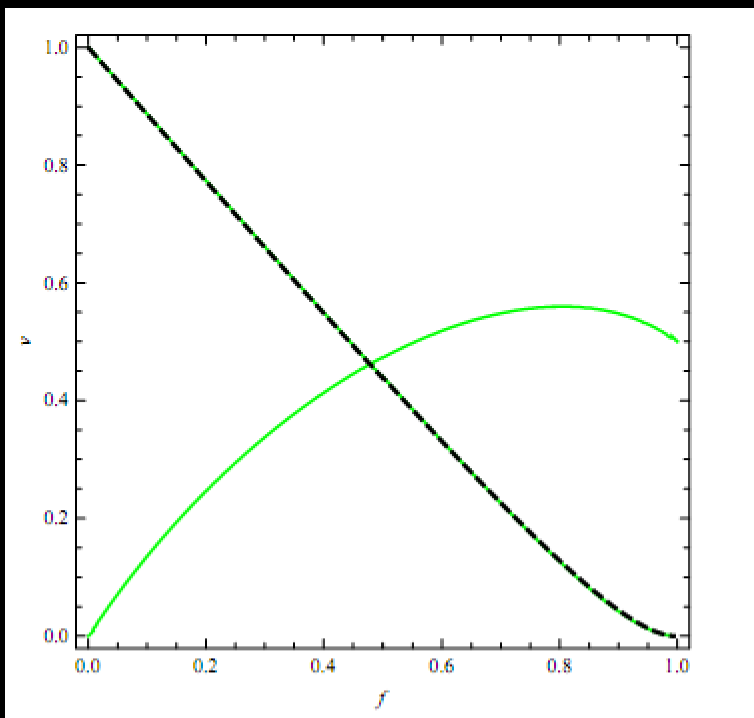
$$\partial_\mu(\sqrt{\tilde{G}}\tilde{G}^{\mu\nu}\partial_\nu\phi_1) = 0.$$

Tortoise coordinate

$$d\rho^* = F \mathcal{L}_W dr,$$

$$F = -\widetilde{G}^{rr}$$

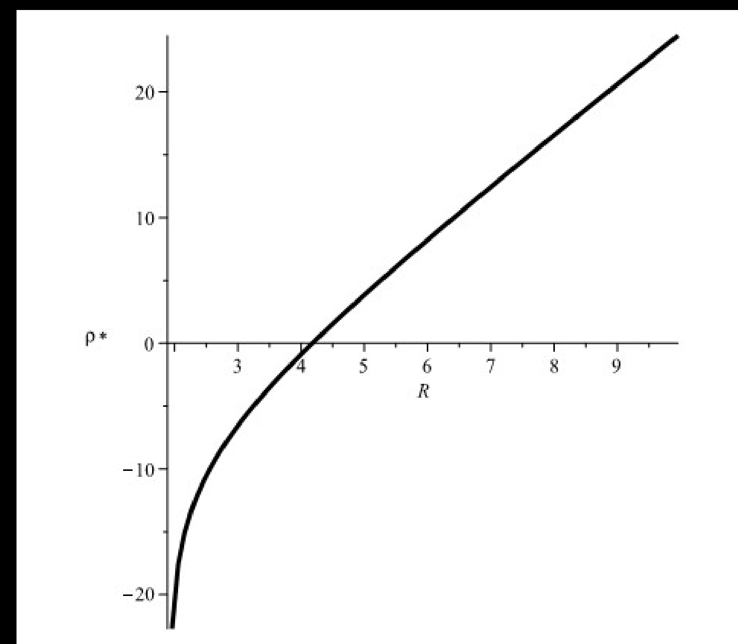
ρ* = ρ*(r) **is calculated numerically, using the parametrization**



$$v = \frac{(f-1)^2}{1 - a_1 f} - a_2 f,$$

$$a_1 = 0.8599,$$

$$a_2 = 0.0003.$$



IST 2015

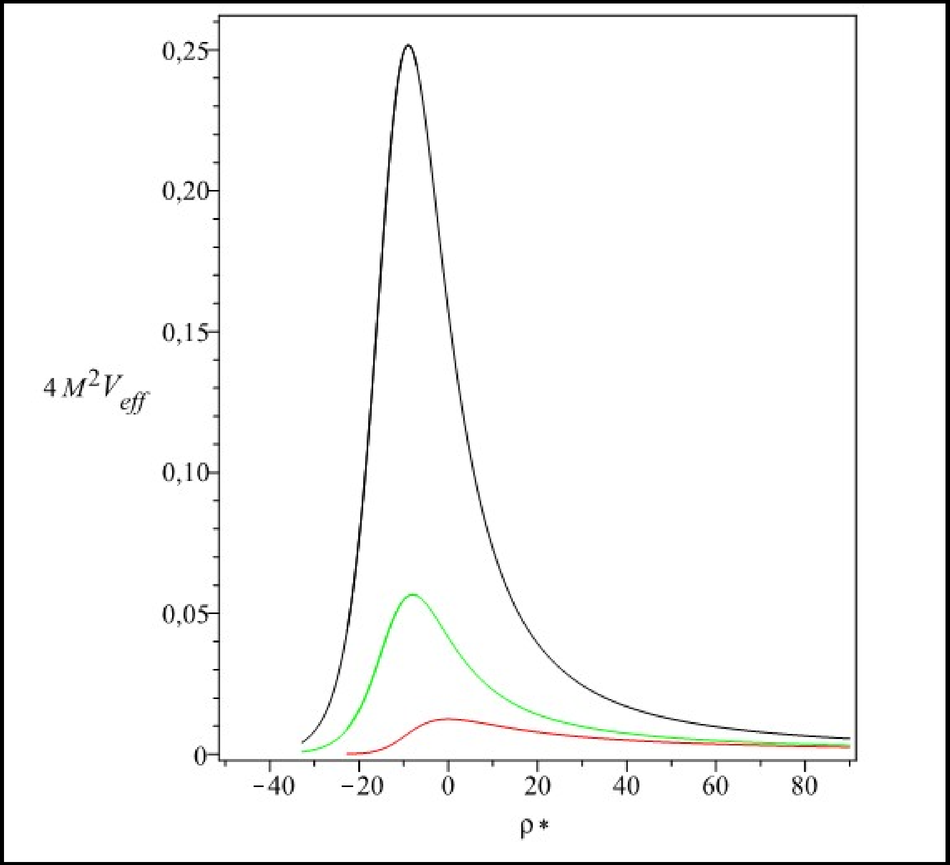
$$\partial_\mu(\sqrt{\tilde{G}}\tilde{G}^{\mu\nu}\partial_\nu\phi_1) = 0.$$

$$\phi_1 = \exp(-i\omega t)\beta(r)Y_{lm}(\theta,\varphi).$$

After a rather long and straightforward calculation,

$$\frac{d^2\beta}{d\rho_*^2} + (\omega^2 - V_{ef}(r))\beta = 0$$

The explicit form of the function *ψ(r)* was needed.



(C. A. Paz Rivasplata, J. M. Salim, SEPB, PRD 2014)

Positivity of the potential is a sufficient condition for linear stability (Wald, 1979).

Conclusions

* The propagation of perturbations of a nonlinear theory is governed by the effective metric, which depends of the nonlinearity of the theory and of the background solution.

* In the case of a test scalar field in stationary accretion on a Schwarzschild bh, the sign of the time derivative of the energy of the perturbations can be determined through a surface integral, that depends only of the sign of the radial derivative of the background solution at r_s.

* Using this integral plus the positivity of the energy of the perturbations, we showed that the model by Frolov is stable under high-energy perts.

* The result coincides with the numerical analysis of the nonlinear stability of particular solutions (Akhoury et al 2011).

* The integral method requires less calculations than the traditional method of the effective potential.

* The method yields a necessary and sufficient condition, while $V_{eff} > 0$ is a sufficient condition.

* Only the sign of the radial derivative of the solution at r_s is needed.

* Near-horizon behaviour of the fields? Generalization to angular momentum? Work in progress with Azucena Paz Rivasplata and Rodrigo Maier).